Operator Theory: Advances and Applications
Vol. 173

Non-spectral Asymptotic Analysis of One-Parameter Operator Semigroups

Eduard Yu. Emel'yanov

Birkhäuser Verlag
Basel · Boston · Berlin

Author:

Eduard Yu. Emel'yanov
Department of Mathematics
Middle East Technial University
06531 Ankara
Turkey
e-mail: eduard@metu.edu.tr

2000 Mathematics Subject Classification 28D05, 37A30, 37A99, 46B03, 46B42, 46L55, 47A10, 47A35, 47B07, 47B60, 47B65, 47D07

Library of Congress Control Number: 2006937469

Bibliographic information published by Die Deutsche Bibliothek
Die Deutsche Bibliothek lists this publication in the Deutsche Nationalbibliografie; detailed bibliographic data is available in the Internet at <http://dnb.ddb.de>.

ISBN 978-3-7643-8095-3 Birkhäuser Verlag, Basel – Boston – Berlin

Part of Springer Science+Business Media
Printed on acid-free paper produced from chlorine-free pulp. TCF ∞
Cover design: Heinz Hiltbrunner, Basel
Printed in Germany
ISBN-10: 3-7643-8095-0
ISBN-13: 978-3-7643-8095-3
e-ISBN-10: 3-7643-8114-0
e-ISBN-13: 978-3-7643-8114-1

9 8 7 6 5 4 3 2 1
www.birkhauser.ch

Contents

Preface

The concept of one-parameter operator semigroups is one of the oldest and most well studied parts of the theory of operators in Banach spaces. It was developed in the middle of the last century as a natural approach to investigation of PDEs and has proved to have many useful and important applications. It is difficult to observe this concept in its entirety, in the sense that there are dozens of textbooks and thousands of research papers directly related to one-parameter operator semigroups. We do not attempt such an overview in this book but devote our attention to some relatively new topics from this theory. Namely, we present and discuss some non-spectral methods that have been developed over the last two decades for the investigation of asymptotic behavior of operator semigroups.

The book has three chapters. The first chapter (with the exception of Section 1.3) contains standard background on operator semigroups. It is addressed to graduate level students. In Section 1.1 we present several basic ergodic theorems for one-parameter semigroups. We discuss here almost periodicity and mean ergodicity of operator semigroups and present such important results as the Jacobs–Deleeuw–Glicksberg splitting theorem and the Eberlein mean ergodic theorem. In Section 1.2 we discuss briefly the spectral theory of C_0-semigroups. We have only one, but quite difficult problem, which is to find a way to make our explanation short enough for a first introduction. All results presented in Sections 1.1 and 1.2 (as well as many others related to C_0-semigroups) can be found in standard textbooks on C_0-semigroups. Then, in Section 1.3 (the content of which is not quite standard), we deal with a very interesting class of one-parameter semigroups — asymptotically finite dimensional semigroups.

Main tools in the investigation of C_0-semigroups are the notions of a generator and analysis of its spectral properties. Such a "spectral" approach to the study of C_0-semigroups is well known and it can be found in standard textbooks and research monographs [13], [41], [48], [57], [80], [89], [104], [130], etc. (see also Section 1.2). However, in many important cases, this approach does not work satisfactorily, particularly in the investigation of asymptotic properties of C_0-semigroups of Markov operators. Recently, essential progress was made in the developing of non-spectral methods in analysis of one-parameter Markov semigroups in L^1-spaces, motivated by applications to probability theory and dynamical systems. These methods and their applications are reflected in the excellent book of Lasota and Mackey [71].

The subject of the second and the third chapters is completely new and not covered in other books, with the exception, that some theorems on Markov operators in commutative L^1-spaces are explained in the above mentioned book [71]. In this part of our book, we give very recent results on non-spectral analysis of asymptotic behavior of positive semigroups and discuss open problems on semigroups of positive operators in ordered Banach spaces.

In the second chapter, we develop some non-spectral methods for the asymptotic analysis of positive one-parameter operator semigroups in ordered Banach spaces. We introduce two classes of ordered Banach spaces which include classical L^p spaces for $1 \leq p < \infty$ as well as preduals of von Neumann algebras. Most results of Section 2.1 are about the asymptotic behavior of positive one-parameter operator semigroups in these spaces. Section 2.2 is devoted to positive semigroups in Banach lattices. In this section we present some theorems on inheritance of asymptotic properties of positive semigroups under the domination, and some theorems concerning the mean ergodicity of positive semigroups. Then in Section 2.3 we discuss several problems on the geometry of Banach spaces, related to one-parameter operator semigroups.

In the third chapter, we investigate positive semigroups in L^1-spaces and in preduals of von Neumann algebras. We study mainly the following two asymptotic problems. The first one is: under what conditions is a one-parameter operator semigroup mean ergodic, almost periodic, or asymptotically stable? The second problem concerns preserving under domination of various asymptotic properties of positive semigroups.

The important aim of the second and the third chapter of the book is an attempt to unify recent results, proofs, and terminology from various sources. We try to present them in a reasonable manner for the potential reader. The author hopes that the bibliography is considerably complete.

The major theorems are usually given with proofs. Only in the case of the proof being too long or involving special methods, the author prefers to send the reader to standard textbooks. At the end of each section we put a supplement, which includes related results, historical notes, exercises, and open problems. In these supplements we usually omit proofs or leave them as exercises for the reader. We assume that the reader is familiar with the basic functional analysis and operator theory, and refer for the more advanced technique to special monographs.

I am indebted to many. I thank Manfred Wolff for many fruitful discussions on this book, my wife Svetlana Gorokhova for helping in preparing of the manuscript, Safak Alpay, Ali Binhadjah, Zafer Ercan, Konstantin Storozhuk, Vladimir Troitsky, and many others for careful reading of the early versions of this book and suggesting of many improvements. I thank the Alexander von Humboldt Foundation for generous support during my stay in 2000–2002 at the University of Tübingen, where the significant part of the work on the second and the third chapters of the book was done.

Ankara, October 2006 Eduard Yu. Emel'yanov

Chapter 1

Elementary theory of one-parameter semigroups

In the first chapter of the book, we give an introduction to the theory of one-parameter operator semigroups. We begin with the splitting theorem of Jacobs–Deleeuw–Glicksberg and the Eberlein mean ergodic theorem for a one-parameter operator semigroup. Then we present the elementary theory of C_0-semigroups and discuss some relations between spectral properties of the generator of a C_0-semigroup and its asymptotic behavior. We follow the standard textbooks [13], [48], [57], [67], [74], [80], [130], and send the reader for other deep and delicious topics of this theory to those books and to [17], [67], [87], [41], [89]. In the last section, we discuss the asymptotically finite-dimensional semigroups. We use frequently well-known results from operator theory and functional analysis, and send the reader to standard textbooks [2], [74], [105], and [130] for them.

1.1 Mean ergodic theorems

In this section we give the main definitions concerning operator semigroups and prove in the one-parameter case the important and rich on applications Jacobs–Deleeuw–Glicksberg's theorem, and we discuss some elementary applications of this theorem. Then we give several basic mean ergodic theorems adapted to one-parameter semigroups. We consider mainly the discrete case and refer the reader for its various generalizations to Krengel's book [67].

1.1.1 Given a real or complex Banach space X, we denote by $\mathcal{L}(X)$ the set of all bounded linear operators in X equipped with the operator norm

$$\|T\| := \sup_{\|x\|\leq 1} \|Tx\|.$$

It is well known that $\mathcal{L}(X) = (\mathcal{L}(X), \|\cdot\|)$ is a *Banach algebra*. In addition to the norm topology, we also consider the *strong* and the *weak* operator topology on $\mathcal{L}(X)$. We shall frequently use the *complexification* of a real vector space and of operators acting on it (see [2] and Exercises 1.1.41, 1.1.42, 1.1.43, and 1.1.44 at the end of this section).

The subject of this book is one-parameter semigroups in $\mathcal{L}(X)$. A non-empty subset $\mathcal{A} \subseteq \mathcal{L}(X)$ is called a *semigroup* if

$$T, S \in A \Rightarrow T \circ S \in \mathcal{A} \quad (\forall\, T, S \in \mathcal{A}).$$

A semigroup $\mathcal{A}$ is called *abelian* if

$$T \circ S = S \circ T \quad (\forall\, T, S \in \mathcal{A}).$$

The definitions of an *idempotent*, *unit*, *nilpotent*, and *inverse element* are standard. Whenever a semigroup has a unit, and every element is invertible, the semigroup becomes a *group*. A subset B of a semigroup $\mathcal{A}$ is called an *ideal* if $\mathcal{A} \circ B \circ \mathcal{A} \subseteq B$.

We shall study mainly operator semigroups indexed by non-negative integers or non-negative reals (we call them *one-parameter semigroups*). Obviously, any one-parameter semigroup is abelian. If a semigroup is indexed by $\mathbb{R}_+$, we always assume that it is *strongly continuous*, that is

$$\lim_{t \to 0} \|T_{s+t}x - T_s x\| = 0 \quad (\forall\, s \geq 0,\, x \in X).$$

Let us use the notation $(T_t)_{t \geq 0}$ for a one-parameter semigroup in the continuous-parameter case, and $(T^n)_{n=1}^{\infty}$ for the *discrete semigroup*, generated by a single operator T. In the following, we also use the notation $\mathcal{T} = (T_t)_{t \in J}$ (where $J = \mathbb{R}_+$ or $J = \mathbb{N} \cup \{0\}$) for any one-parameter semigroup.

We shall assume that any continuously parametered semigroup $\mathcal{T} = (T_t)_{t \geq 0}$ satisfies $T_0 = I$ (where $I = I_X$ denotes the *identity operator* in X). Such a semigroup is called a *C_0-semigroup*. The theory of C_0-semigroups is a very old and rich part of the theory of operators, which possesses many important applications. We give an introduction to this theory in Section 1.2.

A semigroup $\mathcal{T}$ is called *bounded* if

$$M_{\mathcal{T}} := \sup\{\|T\| : T \in \mathcal{T}\} < \infty\,.$$

An operator T is called *power bounded* if the semigroup $(T^n)_{n=0}^{\infty}$ is bounded. T is called *doubly power bounded* if T is invertible and $\sup\{\|T^n\| : n \in \mathbb{Z}\} < \infty$.

Throughout the book, we mainly deal with bounded semigroups. Given a bounded semigroup $\mathcal{T}$, we define an equivalent norm $\|\cdot\|_{\mathcal{T}}$:

$$\|x\|_{\mathcal{T}} := \sup\{\|Tx\| : T \in \mathcal{T}\} \qquad (x \in X).$$

Our semigroup is *contractive* with respect to this new norm, i.e.,

$$\|Tx\|_{\mathcal{T}} \leq \|x\|_{\mathcal{T}} \qquad (\forall\, T \in \mathcal{T}, x \in X).$$

1.1.2 The following class of operators is a starting point for the important notion of almost periodicity. An operator T is called *periodic* if $T^{p+1} = T$ for some $p \in \mathbb{N}$. If T is a periodic operator, then

$$\mathrm{per}_T := \inf\{m \geq 1 : T = T^{m+1}\}$$

is called the *period* of T. In this case, T^{per_T} is a projection. Any periodic operator T, obviously, has the property that any *orbit* $\{T^n x\}_{n=0}^{\infty}$ is a finite and therefore compact subset of X. This observation leads to the following definition.

Definition 1.1.1. A one-parameter operator semigroup $\mathcal{T}$ in a Banach space X is called (*weakly*) *almost periodic* whenever the *orbit* $\mathcal{T}x = \{T_t x\}_{t \in J}$ is (weakly) precompact for any $x \in X$. An operator $T \in \mathcal{L}(X)$ is called (*weakly*) *almost periodic* if the semigroup $(T^n)_{n=0}^{\infty}$ is (weakly) almost periodic.

It is an easy exercise to show that $\mathcal{T}$ is (weakly) almost periodic iff, for every $x \in X$ and every sequence $(t_n)_{n=1}^{\infty}$ in J converging to ∞, there is a subsequence $(t_{n_k})_{k=1}^{\infty}$ of $(t_n)_{n=1}^{\infty}$ such that the limit

$$\left(w - \lim_{k\to\infty} T_{t_{n_k}} x\right) \qquad \|\cdot\| - \lim_{k\to\infty} T_{t_{n_k}} x$$

exists. By the uniform boundedness principle, any weakly almost periodic semigroup is bounded. In many cases, the converse is also true. For example, since the closed unit ball of a reflexive Banach space is weakly compact, we have (due to the Eberlein–Smulian theorem) the following result.

Theorem 1.1.2. *Any bounded one-parameter semigroup $\mathcal{T}$ in a reflexive Banach space is weakly almost periodic.* □

A one-parameter semigroup $\mathcal{T} = (T_t)_{t \in J}$ in X is called *strongly stable* if $\|\cdot\|\text{-}\lim_{t\to\infty} T_t x$ exists for all $x \in X$. An operator $T \in \mathcal{L}(X)$ is called *strongly stable* whenever the semigroup $\mathcal{T} = (T^n)_{n=0}^{\infty}$ is strongly stable. Of course, any strongly stable semigroup is almost periodic. Remark that some authors use the term "(strongly) stable semigroup" if $\lim_{t\to\infty} \|T_t x\| = 0$ for all $x \in X$.

We say that an operator $T \in \mathcal{L}(X)$ has *finite rank* if $\dim T(X) < \infty$. In this case we denote $\dim T(X)$ by $\mathrm{rank}(T)$. An operator T is called *asymptotically periodic* whenever there exists a periodic operator Q of finite rank such that the sequence $(T^n - Q^n)_{n=0}^{\infty}$ converges to 0 in the strong operator topology. In this case, $\mathrm{aper}_T := \mathrm{per}_Q$ is called the *asymptotic period* of T. Any asymptotically periodic operator is almost periodic.

In Chapters 2 and 3, we shall study in detail strongly stable and asymptotically periodic semigroups of positive operators.

1.1.3 The next proposition reduces the notion of almost periodicity of a one-parameter semigroup to its special case, namely, almost periodicity of a single operator.

Proposition 1.1.3. *Given a bounded one-parameter semigroup $\mathcal{T} = (T_t)_{t\in J}$ such that the operator T_τ is (weakly) almost periodic for some $\tau \in J$, then the set*

$$\{T_{n\tau}x : n \in \mathbb{N}, x \in K\}$$

is (weakly) conditionally compact for any compact $K \subseteq X$. In particular, the semigroup $\mathcal{T}$ is (weakly) almost periodic.

Proof. By the uniform boundedness principle, $M_{\mathcal{T}} = \sup_{t\in J} \|T_t\| < \infty$. Take an $\varepsilon > 0$ and let $\{x_i\}_{i=1}^k$ be a finite ε-net for K. The set

$$L_\varepsilon := \{T_{n\tau}x_i : n \in \mathbb{N}, i = 1, \ldots, k\}$$

is conditionally (weakly) compact due to (weakly) almost periodicity of $(T_\tau^n)_{n=1}^\infty$ and satisfies

$$\{T_{n\tau}x : n \in \mathbb{N}, x \in K\} \subseteq L_\varepsilon + M_{\mathcal{T}} \cdot \varepsilon \cdot B_X.$$

Thus we have obtained that the set

$$\{T_{n\tau}x : n \in \mathbb{N}, x \in K\}$$

can be arbitrarily well approximated in norm by conditionally (weakly) compact sets. Hence the set $\{T_{n\tau}x : n \in \mathbb{N}, x \in K\}$ is also conditionally (weakly) compact.

Take an arbitrary $x \in X$. Then the set

$$K = \{T_t x : 0 \le t \le \tau\}$$

is compact (it follows from the strong continuity if $\mathcal{T} = (T_t)_{t\ge 0}$, and it is trivial if $\mathcal{T}$ is discrete), and

$$\{T_t x : t \in J\} = \{T_{n\tau}x : n \in \mathbb{N}, x \in K\}$$

is conditionally (weakly) compact, as it has been shown. Hence $\mathcal{T}$ is (weakly) almost periodic. □

1.1.4 Now we state one of the most powerful results of operator theory: the splitting theorem of Jacobs–Deleeuw–Glicksberg [75]. This theorem is very useful in the investigation of asymptotic behavior of one-parameter semigroups. Though it is about general abelian semigroups, we shall use it in our book for one-parameter semigroups only.

Let us recall that, for a given semigroup J, a multiplicative homomorphism $\alpha : J \to \Gamma$, where $\Gamma := \{\lambda \in \mathbb{C} : |\lambda| = 1\}$ is the *unit circle* in $\mathbb{C}$, is called a *character*.

Theorem 1.1.4 (Jacobs–Deleeuw–Glicksberg). *Given a weakly almost periodic semigroup $\mathcal{T} = (T_\tau)_{\tau\in J}$ in a Banach space X, then X can be decomposed into the direct sum $X = X_{fl}(\mathcal{T}) \oplus X_r(\mathcal{T})$, where*

$$X_{fl}(\mathcal{T}) = \{x \in X : 0 \in \text{w-cl}\{T_t x\}_{t\in J}\}, \ \textit{and}$$

$$X_r(\mathcal{T}) = \{x \in X : y \in \text{w-cl}\{T_t x\}_{t\in J} \Rightarrow x \in \text{w-cl}\{T_t y\}_{t\in J}\}.$$

Moreover, the restriction of the closure of $\mathcal{T}$ in the weak operator topology to $X_r(\mathcal{T})$ is a group, and

$$X_r(\mathcal{T}) = \overline{\text{span}}\{x \in X : \exists \text{ a character } \alpha : J \to \Gamma \text{ s.t. } \forall t \in J\ T_t x = \alpha(t)x\}. \quad (1.1)$$

If we assume additionally that $\mathcal{T}$ is almost periodic, then

$$X_{fl}(\mathcal{T}) = \{x \in X : \lim_{t\to\infty} \|T_t x\| = 0\}. \quad (1.2)$$

This theorem possesses many important applications in operator theory and PDEs. Some of them will be discussed in Sections 1.2, 2.2, and 3.3, and we refer for many others to [13], [41], [67], and [80]. Below we give the proof of this theorem with one exception. We do not give the proof of the inclusion "$\subseteq$" of (1.1), since this part of the proof involves some technique that we shall not explain in the book.

Before we prove Theorem 1.1.4, we need two lemmas. The first lemma is about the weak operator topology in $\mathcal{L}(X)$. Recall that the weak operator topology is defined by the convergence "$T_\alpha \to T$" iff

$$\lim_{\alpha\to\infty} \langle (T_\alpha - T)x, x' \rangle = 0 \quad (\forall\ x \in X,\ x' \in X^*).$$

For short, we prefer to use the term *wo-topology*. We denote by $\bar{\mathcal{A}} = \text{wo-cl}(\mathcal{A})$ the closure of a set $\mathcal{A} \subseteq \mathcal{L}(X)$ in the wo-topology.

Lemma 1.1.5. *A subset $\mathcal{A} \subseteq \mathcal{L}(X)$ is relatively compact in the wo-topology iff its* orbit $\mathcal{A}x := \{Tx : T \in \mathcal{A}\}$ *is relatively weakly compact for any $x \in X$. Moreover,*

$$\bar{\mathcal{A}}x = \text{w-cl}(\mathcal{A}x) \quad (\forall x \in X). \quad (1.3)$$

Proof. If $\mathcal{A}$ is relatively wo-compact then $\bar{\mathcal{A}}$ is wo-compact, and $\bar{\mathcal{A}}x$ is weakly compact for any x as the continuous image of a compact set.

Conversely, assume that $\mathcal{A}x$ is relatively weakly compact for all x. For each x, let $A_x = \text{w-cl}(\mathcal{A}x)$ and let $A = \prod_{x\in X} A_x$ with the product topology. As each A_x is compact, A is compact by the Tychonoff product theorem. Now let $(T_\alpha)_\alpha$ be an arbitrary net in $\mathcal{A}$. Then $((T_\alpha x)_{x\in X})_\alpha$ is a net in A and, by the compactness of A, there exists a subnet $((T_\beta x)_{x\in X})_\beta$ converging in A. This means that the weak limit $\text{w-}\lim_\beta T_\beta x$ exists for all x. Define an operator T by $Tx = \text{w-}\lim_\beta T_\beta x$. Clearly, T is linear, and $T = \text{wo-}\lim_\beta T_\beta \in \bar{\mathcal{A}}$. As $(T_\alpha)_\alpha$ is an arbitrary net, we have proved that $\mathcal{A}$ is relatively wo-compact.

In the last assertion, the inclusion $\bar{\mathcal{A}}x \supseteq \text{w-cl}(\mathcal{A}x)$ follows from the weak compactness of $\bar{\mathcal{A}}x$, and the inclusion $\bar{\mathcal{A}}x \subseteq \text{w-cl}(\mathcal{A}x)$ follows from the definition of the wo-topology. □

Lemma 1.1.6 (Kernel lemma). *Let $\mathcal{J}$ be a compact abelian semigroup in which the multiplication is separately continuous (i.e., $T_\alpha \to T$ implies $S \circ T_\alpha \to S \circ T$ for all $S \in \mathcal{J}$ and $(T_\alpha)_\alpha \subseteq \mathcal{J}$). Then $\mathcal{J}$ contains a unique minimal ideal $\mathcal{K}$ which is called the* kernel *of $\mathcal{J}$, and*

$$\mathcal{K} = \bigcap_{S \in \mathcal{J}} S \circ \mathcal{J}. \tag{1.4}$$

Moreover, $\mathcal{K}$ is a group, and if P is its unit, then $\mathcal{K} = P \circ \mathcal{J}$.

Proof. If $J_1, \dots, J_k$ are ideals in $\mathcal{J}$ (i.e., $A \circ B \in J_i$ for all $A \in \mathcal{J}$, $B \in J_i$), their product $J_1 \circ J_2 \circ \cdots \circ J_k$ is an ideal which is not empty and is contained in their intersection, since $\mathcal{J}$ is abelian. Consequently, the family of closed ideals of $\mathcal{J}$ has the finite intersection property. As $\mathcal{J}$ is compact, the intersection $\mathcal{K}$ of all closed ideals is non-empty.

If J is any ideal and $S \in J$, then $S \circ \mathcal{J}$ is a closed ideal contained in J due to compactness of $\mathcal{J}$ and to the separate continuity of the multiplication. Hence $\mathcal{K}$ is the intersection of all ideals. As each $S \circ \mathcal{J}$ is an ideal and each ideal contains an ideal of the form $S \circ \mathcal{J}$, (1.4) follows. Since $\mathcal{K}$ is the intersection of a family of ideals, $\mathcal{K}$ is an ideal. Clearly $\mathcal{K}$ is minimal and unique.

For any $T \in \mathcal{K}$, the set $T \circ \mathcal{K}$ is an ideal contained in $\mathcal{K}$ and, therefore, is equal to $\mathcal{K}$ by the minimality of $\mathcal{K}$. So there exists $P \in \mathcal{K}$ such that $T \circ P = T$. By $\mathcal{K} \circ T = T \circ \mathcal{K} = \mathcal{K}$, for any $S \in \mathcal{K}$ there exists $R \in \mathcal{K}$ with $R \circ T = S$. Hence

$$S \circ P = R \circ T \circ P = R \circ T = S,$$

which shows that $S \circ P = S$ holds for all $S \in \mathcal{K}$. Then P is a unit. Finally, $T \circ \mathcal{K} = \mathcal{K}$ and $P \in \mathcal{K}$ implies existence of the inverse of T in $\mathcal{K}$.

The last assertion follows from $P \circ T \subseteq \mathcal{K} \circ \mathcal{J} \subseteq \mathcal{K}$ and (1.4). □

It is an easy exercise to show that the multiplication in $\mathcal{L}(X)$ is separately continuous in the wo-topology, i.e.,

$$T_\alpha \to T \text{ in wo-topology } \Rightarrow \; T_\alpha \circ S \to T \circ S \;\;\&\;\; S \circ T_\alpha \to S \circ T \text{ in wo-topology}$$

for any net $(T_\alpha)_\alpha \subseteq \mathcal{L}(X)$ and for any $S \in \mathcal{L}(X)$. However, the multiplication in $\mathcal{L}(X)$ is not jointly continuous in general.

1.1.5 Now we are able to prove Theorem 1.1.4. As well as in the proofs of Lemmas 1.1.5 and 1.1.6 above, we shall follow Krengel [67].

We call elements of $X_{fl}(\mathcal{T})$ *flight* vectors, and elements of $X_r(\mathcal{T})$ *reversible* vectors. An element $x \neq 0$ of X is said to be a *unimodular eigenvector* of a semigroup $\mathcal{T} = (T_t)_{t \in J}$ if there is a character $\alpha : J \to \Gamma$ such that $T_t x = \alpha(t) x$ for all $t \in J$. Denote by $X_{ue}(\mathcal{T})$ the closure of the subspace of X spanned by all unimodular eigenvectors.

If $\mathcal{A}$ is a semigroup in $\mathcal{L}(X)$, then $\mathcal{A} \circ \mathcal{A} \subseteq \mathcal{A}$, and hence $\mathcal{A} \circ \bar{\mathcal{A}} \subseteq \bar{\mathcal{A}}$ by the separate continuity. Applying the separate continuity again, we obtain $\bar{\mathcal{A}} \circ \bar{\mathcal{A}} \subseteq \bar{\mathcal{A}}$, i.e., $\bar{\mathcal{A}}$ is a semigroup. Moreover, if $\mathcal{A}$ is abelian then $\bar{\mathcal{A}}$ is, obviously, abelian.

Proof of Theorem 1.1.4. The wo-closure $\bar{\mathcal{T}}$ of $\mathcal{T}$ is a wo-compact abelian *semi-topological semigroup* in $\mathcal{L}(X)$ (i.e., the multiplication in $\bar{\mathcal{T}}$ is separately continuous) accordingly to Lemma 1.1.5. Denote by P the unit of the kernel $\mathcal{K}$ of $\bar{\mathcal{T}}$, existence of which is provided by Lemma 1.1.6.

For any $T \in \bar{\mathcal{T}}$, there exists $S \in \mathcal{K}$ with $S \circ T \circ P = P$, because $\mathcal{K}$ is a group and $T \circ P \in \mathcal{K}$. If $x = Py$ for some y, then

$$x = S \circ T \circ Py = S \circ Tx. \tag{1.5}$$

By (1.3), $\bar{\mathcal{T}}x = \text{w-cl}(\mathcal{T}x)$. So (1.5) implies that x is reversible. If $x \in X_r(\mathcal{T})$ then, again by Lemma 1.1.5, there exists $R \in \bar{\mathcal{T}}$ with $x = R \circ Px$. Since $\mathcal{T}$ is abelian, $x = P \circ Rx$, and so $x \in P(X)$. This shows that $P(X) = X_r(\mathcal{T})$.

By (1.3), all elements of $\ker P = \{x : Px = 0\}$ are flight vectors. On the other hand, if x is a flight vector, there exists $R \in \bar{\mathcal{T}}$ with $Rx = 0$. Hence $P \circ Rx = 0$. As $P \circ R$ belongs to $\mathcal{K}$, there exists S with $S \circ P \circ R = P$. Thus, $Px = S \circ P \circ Rx = 0$. This shows that $\ker P = X_{fl}(\mathcal{T})$.

Obviously, $\ker P = (I - P)X$. As any x can be written in the form

$$x = Px + (I - P)x,$$

we find $x = x_1 + x_2$ with $x_1 \in X_r(\mathcal{T})$, $x_2 \in X_{fl}(\mathcal{T})$. Conversely, if x is of the form

$$x = x_1 + x_2 \quad (x_1 \in X_r(\mathcal{T}),\ x_2 \in X_{fl}(\mathcal{T})),$$

we have $Px = Px_1$ and $x_1 = Py_1$ for some $y_1 \in X$, and hence

$$x_1 = Py_1 = P^2 y_1 = Px.$$

Therefore, the splitting is unique.

The group property of $\bar{\mathcal{T}}|_{X_r}$ follows from the group property of $\mathcal{K}$.

Now, to finish the proof, we have to show equalities (1.1) and (1.2). The equality (1.1) says that $X_r(\mathcal{T}) = X_{ue}(\mathcal{T})$. The inclusion

$$X_{ue}(\mathcal{T}) \subseteq X_r(\mathcal{T})$$

is easy: For $x \in X_{ue}(\mathcal{T})$ and $T \in \bar{\mathcal{T}}$, the identity $T^n x = \alpha(T)^n x$ shows the existence of a sequence $(n_i)_{i=1}^{\infty}$ with $T^{n_i} x \to x$. It follows that, for any $h_1, \dots, h_m \in X^*$ and $\varepsilon > 0$, the set

$$\{S \in \bar{\mathcal{T}} : |\langle S \circ Tx - x, h_i \rangle| \le \varepsilon, i = 1, \dots, m\}$$

is non-empty, and any finite intersection of such sets is non-empty. By the compactness of $\bar{\mathcal{T}}$ in the wo-topology, the intersection of all sets of this type is non-empty. Therefore there is $S \in \bar{\mathcal{T}}$ such that $|\langle S \circ Tx - x, h \rangle| \le \varepsilon$ holds for all $\varepsilon > 0$ and all $h \in X^*$. But then $S \circ Tx = x$. As $T \in \bar{\mathcal{T}}$ was arbitrary, x is reversible.

The proof of the inclusion $X_r(\mathcal{T}) \subseteq X_{ue}(\mathcal{T})$ is more technical and depends on some results of abstract harmonic analysis and semi-topological groups. We omit this part of the proof and send the reader to [67, pp. 106–107] for it.

To show (1.1), it is obviously enough to check that if the semigroup $\mathcal{T}$ is almost periodic and $0 \in \text{w-cl}\{T_t x\}_{t\in J}$, then $\lim\limits_{t\to\infty} \|T_t x\| = 0$. Indeed, $0 \in \text{w-cl}\{T_t x\}_{t\in J}$ implies existence of a net $(t_\alpha)_\alpha$ such that $T_{t_\alpha} x \to 0$ weakly. By the relatively norm compactness of $\{T_{t_\alpha} x\}_\alpha$, there is a subnet $(t_{\alpha_\beta})_\beta$ such that $T_{t_{\alpha_\beta}} x \to x_0$ in norm for some $x_0 \in X$ as $\beta \to \infty$. Since $T_{t_{\alpha_\beta}} x \to 0$ weakly, we obtain $x_0 = 0$. Then $\lim\limits_{\beta\to\infty} \|T_{t_{\alpha_\beta}} x\| = 0$, and hence

$$\inf\{\|T_t x\| : t \in J\} = 0\,. \tag{1.6}$$

The boundedness of $\mathcal{T}$ and (1.6) imply $\lim\limits_{t\to\infty} \|T_t x\| = 0$, which is required. $\square$

1.1.6 Given a one-parameter semigroup $\mathcal{T}$ in $\mathcal{L}(X)$, denote the *Cesàro averages* (*means*) of $\mathcal{T}$ by

$$\mathcal{A}_\tau^{\mathcal{T}} = \mathcal{A}_\tau^{T} := \frac{1}{\tau} \sum_{k=0}^{\tau-1} T^k \qquad \text{(whenever } \mathcal{T} = (T^n)_{n=0}^\infty\text{)},$$

$$\mathcal{A}_\tau^{\mathcal{T}} := \frac{1}{\tau} \int_0^\tau T_t dt \qquad \text{(whenever } \mathcal{T} = (T_t)_{t\geq 0}\text{)}.$$

The integral above is taken with respect to the strong topology on $\mathcal{L}(X)$. A one-parameter semigroup $\mathcal{T}$ in $\mathcal{L}(X)$ is called *Cesàro bounded* if

$$\sup_\tau \|\mathcal{A}_\tau^{\mathcal{T}}\| < \infty.$$

The following theorem gives some important conditions for the norm convergence of $\mathcal{A}_\tau^{\mathcal{T}} x$ as $\tau \to \infty$. For interesting historical remarks on this and other mean ergodic theorems, we refer to the survey of Shaw [116].

Theorem 1.1.7 (Eberlein). *Let $\mathcal{T}$ be a one-parameter Cesàro bounded semigroup in a Banach space X. Then, for any $x \in X$ satisfying*

$$\lim_{t\to\infty} \|t^{-1} T_t x\| = 0 \tag{1.7}$$

and for any $y \in X$, the following conditions are equivalent:

(i) $\mathcal{T} y = y$ *and* $y \in \overline{\text{co}}\{T_t x : t \in J\}$;

(ii) $y = \lim\limits_{t\to\infty} \mathcal{A}_t^{\mathcal{T}} x$;

(iii) y *is a weak cluster point of the net* $(\mathcal{A}_t^{\mathcal{T}} x)_{t\in J}$.

Proof. We consider only the discrete case $\mathcal{T} = (T^n)_{n=0}^\infty$ and follow Krengel [67, 2.2.1] in the proof.

(i) $\Rightarrow$ (ii): Set $M := \sup_n \|\mathcal{A}_n^T\|$. For $\varepsilon > 0$, (i) implies $\|y - Sx\| \leq \varepsilon$ for an appropriate $S \in \mathrm{co}\{T^0, T^1, T^2, \ldots\}$. For any k and n, we have

$$T^k \circ \mathcal{A}_n^T - \mathcal{A}_n^T = n^{-1} \sum_{i=0}^{n-1} T^{k+i} - n^{-1} \sum_{i=0}^{n-1} T^i.$$

By using (1.7), we see that

$$\|T^k \circ \mathcal{A}_n^T x - \mathcal{A}_n^T x\| \leq n^{-1} \sum_{j=0}^{k-1} \|T^j x\| + n^{-1} \|T^n x\| \sum_{j=0}^{k-1} \|T^j\| \leq \varepsilon$$

holds for large enough $n \geq k$. Since S is a convex combination of finitely many T^k, we may find $n(\varepsilon)$ such that $\|S \circ \mathcal{A}_n^T x - \mathcal{A}_n^T x\| \leq \varepsilon$ holds for $n \geq n(\varepsilon)$. As y is a fixed point of T, we have $\mathcal{A}_n^T y = y$ for all n. For $n \geq n(\varepsilon)$, we therefore obtain

$$\begin{aligned} \|y - \mathcal{A}_n^T x\| &\leq \|\mathcal{A}_n^T(y - Sx)\| + \|S \circ \mathcal{A}_n^T x - \mathcal{A}_n^T x\| \\ &\leq M\varepsilon + \varepsilon\,, \end{aligned}$$

which means that $\lim_{n\to\infty} \|y - \mathcal{A}_n^T x\| = 0$.

The implication (ii) $\Rightarrow$ (iii) is trivial.

(iii) $\Rightarrow$ (i): By the Mazur theorem, any closed convex subset of X is weakly closed. Thus the weak cluster point y of the set $\{\mathcal{A}_n^T x\}_{n=0}^\infty \subseteq \mathrm{co}\{T^n x\}_{n=0}^\infty$ belongs to $\overline{\mathrm{co}}\,\{T^n x\}_{n=0}^\infty$.

To prove $y = Ty$, take any $h \in X^*$ and $\varepsilon > 0$. By the argument used above, we know that $\|T \circ \mathcal{A}_n^T x - \mathcal{A}_n^T x\| \to 0$. For a large enough n,

$$|\langle \mathcal{A}_n^T x, h\rangle - \langle T \circ \mathcal{A}_n^T x, h\rangle| = |\langle T \circ \mathcal{A}_n^T x - \mathcal{A}_n^T x, h\rangle| \leq \varepsilon\,.$$

As y is a weak cluster point of $\{\mathcal{A}_n^T x\}_{n=0}^\infty$, there exists an arbitrarily large n with

$$|\langle y, h\rangle - \langle \mathcal{A}_n^T x, h\rangle| \leq \varepsilon$$

and

$$|\langle T \circ \mathcal{A}_n^T x, h\rangle - \langle Ty, h\rangle| = |\langle y, T^* h\rangle - \langle \mathcal{A}_n^T x, T^* h\rangle| \leq \varepsilon\,.$$

Combining these three estimates, we arrive at $|\langle y, h\rangle - \langle Ty, h\rangle| \leq 3\varepsilon$. The arbitrariness of ε and h implies $Ty = y$. □

Definition 1.1.8. A one-parameter semigroup $\mathcal{T}$ is called *mean ergodic* if the norm limit $\lim_{\tau\to\infty} \mathcal{A}_\tau^{\mathcal{T}} x$ exists for all $x \in X$.

We call an operator T *mean ergodic* whenever the semigroup $\mathcal{T} = (T^n)_{n=1}^\infty$ is mean ergodic. Remark that any one-parameter semigroup which is mean ergodic is Cesàro bounded and satisfies (1.7). This assertion is left to the reader as an exercise. Theorem 1.1.7 says that for proving mean ergodicity of $\mathcal{T}$ it is enough to check that the set $\{\mathcal{A}_t^{\mathcal{T}} x : t \in J\}$ has a weak cluster point for each $x \in X$. In particular, this means that we can reply the norm limit in Definition 1.1.8 by the weak one.

1.1.7 In the case $\mathcal{T} = (T^n)_{n=1}^\infty$, we use the following notation:

$$X_{me}(T) := \{x \in X : \exists \lim_{n\to\infty} \mathcal{A}_n^T x\}, \qquad N(T) := (I-T)X,$$

$$\mathrm{Fix}(T) := \{x \in X : Tx = x\}, \qquad \mathrm{Fix}(T^*) := \{y \in X^* : T^*y = y\}.$$

Clearly, all these sets are linear subspaces of X and X^*. The next theorem gives the relations between them.

Theorem 1.1.9 (Yosida). *Let T be a Cesàro bounded operator in a Banach space X which satisfies (1.7) for all $x \in X$. Then $X_{me}(T) = \mathrm{Fix}(T) \oplus \overline{N(T)}$, and the operator $P : X_{me}(T) \to X$, which is defined as*

$$Px := \lim_{n\to\infty} \mathcal{A}_n^T x \qquad (x \in X_{me}(T)),$$

is a projection of $X_{me}(T)$ onto $\mathrm{Fix}(T)$ satisfying $P = T \circ P = P \circ T$. Moreover, for any $x \in X$, the assertions:

(i) $\lim_{n\to\infty} \mathcal{A}_n^T x = 0$,

(ii) $\langle x, h\rangle = 0$ *for all* $h \in \mathrm{Fix}(T^*)$,

(iii) $x \in \overline{N(T)}$,

are equivalent.

Proof. Clearly $\mathrm{Fix}(T)$ and $\overline{N(T)}$ are closed subspaces in X. Let us verify

$$\mathrm{Fix}(T) \cap \overline{N(T)} = \{0\}.$$

For $\varepsilon > 0$ and $z \in \mathrm{Fix}(T) \cap \overline{N(T)}$, there exists u with $\|z - (u - Tu)\| \le \varepsilon$. Hence

$$\|\mathcal{A}_n^T (z - (u - Tu))\| \le M\varepsilon\,.$$

Using $\mathcal{A}_n^T z = z$ and $\mathcal{A}_n^T (u - Tu) \to 0$, we find $\|z\| \le M\varepsilon + \varepsilon$, where $M = \sup_{n\ge 0} \|\mathcal{A}_n^T\|$. Therefore $z = 0$. A similar argument can be used to show that $\mathrm{Fix}(T^*) \cap \overline{N(T^*)} = \{0\}$.

(i) $\Rightarrow$ (ii): The condition $h \in \mathrm{Fix}(T^*)$ implies $h = \mathcal{A}_n^{T^*} h$ for all n, and hence

$$\langle x, h\rangle = \langle x, \mathcal{A}_n^{T^*} h\rangle = \langle \mathcal{A}_n^T x, h\rangle \to 0.$$

(ii) $\Rightarrow$ (iii): If $x \notin \overline{N(T)}$, there exists, by the Hahn–Banach theorem, $h \in X_*$ with $\langle x, h\rangle \ne 0$ and $\langle y, h\rangle = 0$ for all $y \in \overline{N(T)}$. In particular, $\langle u - Tu, h\rangle = 0$ for all $u \in X$. Hence $\langle u, h - T^*h\rangle = 0$ for all $u \in X$. This implies $h \in \mathrm{Fix}(T^*)$ and, therefore, by *(ii)*, $\langle x, h\rangle = 0$ which is a contradiction.

(iii) $\Rightarrow$ (i): Let $x \in \overline{N(T)}$. For any $u \in X$, we have

$$\mathcal{A}_n^T (u - Tu) = n^{-1}(u - T^n u) \to 0.$$

Thus, all $z \in N(T)$ satisfy $\|\mathcal{A}_n^T z\| \to 0$. The set of all $z \in X$ with the property $\|\mathcal{A}_n^T z\| \to 0$ is closed because T is Cesàro bounded. Thus $\lim_{n\to\infty} \mathcal{A}_n^T x = 0$.

For $x \in X_{me}$, Theorem 1.1.7 implies $Px \in \mathrm{Fix}(T)$. Thus $z = x - Px$ satisfies (i). We have proved $X_{me} \subseteq \mathrm{Fix}(T) \oplus \overline{N(T)}$. As $\mathrm{Fix}(T) \subseteq X_{me}$ is trivial, the opposite inclusion follows from (iii) $\Rightarrow$ (i).

As P is the projection of $\mathrm{Fix}(T) \oplus \overline{N(T)}$ on $\mathrm{Fix}(T)$ and the elements of $\mathrm{Fix}(T)$ are fixed under T, the identity $P = T \circ P$ is clear. $P = P \circ T$ follows from $\mathcal{A}_n^T(x - Tx) \to 0$. ☐

Corollary 1.1.10. *Let T be a Cesàro bounded operator in X which satisfies* (1.7). *Then T is mean ergodic if and only if $X = \mathrm{Fix}(T) \oplus \overline{N(T)}$.* ☐

1.1.8 The following criterion of Sine [117] for mean ergodicity is very useful, despite its simplicity, and we apply it often below.

Theorem 1.1.11 (Sine). *Let T be a Cesàro bounded operator in X which satisfies* (1.7). *Then T is mean ergodic if and only if* $\mathrm{Fix}(T)$ *separates* $\mathrm{Fix}(T^*)$.

Proof. Let T be mean ergodic, and let $P : X \to \mathrm{Fix}(T)$ be the correspondent mean ergodic projection. If $0 \neq h \in \mathrm{Fix}(T^*)$, there exists $x \in X$ with $\langle x, h\rangle \neq 0$. Now Px belongs to $\mathrm{Fix}(T)$ and separates h and 0, because of

$$\begin{aligned}\langle Px, h\rangle = \lim_{n\to\infty} \langle A_n^T x, h\rangle &= \lim_{n\to\infty} \langle x, A_n^{T^*} h\rangle \\ &= \langle x, h\rangle \neq 0.\end{aligned}$$

Assume that $\mathrm{Fix}(T)$ separates $\mathrm{Fix}(T^*)$. If $\mathrm{Fix}(T) \oplus \overline{N(T)} \neq X$, there exists

$$0 \neq h \in X^*$$

with $\langle y, h\rangle = 0$ for all $y \in \mathrm{Fix}(T) \oplus \overline{N(T)}$. In particular, $\langle x - Tx, h\rangle = 0$ for all $x \in X$ shows $h \in \mathrm{Fix}(T^*)$. As $\mathrm{Fix}(T)$ separates h and 0, there exists $y \in \mathrm{Fix}(T)$ with $\langle y, h\rangle \neq 0$, a contradiction. ☐

1.1.9 Let us give some further conditions for the mean ergodicity. We begin with the discrete case.

Theorem 1.1.12. *Let T^m be mean ergodic for some m, then T is mean ergodic.*

Proof. The assertion follows immediately from the equality

$$\mathcal{A}_{nm}^T = \frac{1}{m}\sum_{k=0}^{m-1} T^k \mathcal{A}_n^{T^m} \qquad (\forall n, m \in \mathbb{N})$$

and from the Eberlein theorem. ☐

Theorem 1.1.13. *Given a bounded C_0-semigroup $\mathcal{T} = (T_t)_{t\geq 0}$ such that an operator T_ξ is mean ergodic for some $\xi \in J$, then $\mathcal{T}$ is mean ergodic.*

Proof. Let T_ξ be mean ergodic for some $\xi \in J$. Then $T_{\xi/k}$ is mean ergodic for all $k \in \mathbb{N}$, as it has been shown above in Theorem 1.1.12. Given $x \in X$ and $\varepsilon > 0$, then

$$\|\mathcal{A}^{\mathcal{T}}_{n\xi/m}x - \mathcal{A}^{T_{\xi/m}}_n x\| \leq \varepsilon/2 \qquad (\forall n \in \mathbb{N})$$

for any $m \geq m_1$. On the other hand, there exists a large enough $m_2 \geq m_1$ such that for any $t \in \mathbb{R}_+$ there exists n satisfying

$$\|\mathcal{A}^{\mathcal{T}}_t x - \mathcal{A}^{\mathcal{T}}_{n\xi/m_2}x\| \leq \varepsilon/2.$$

Combining these two inequalities, we obtain

$$\{\mathcal{A}^{\mathcal{T}}_t x\}_{t\in J} \subseteq \{\mathcal{A}^{T_{\xi/m_2}}_n x : n \in \mathbb{N}\} + \varepsilon \cdot B_X.$$

Hence the set $\{\mathcal{A}^{\mathcal{T}}_t x\}_{t\geq 0}$ can be arbitrarily well approximated by relatively weakly compact sets and, due to Theorem 1.1.7, the semigroup $\mathcal{T}$ is mean ergodic. □

Related Results and Notes

1.1.10 The origin of Lemma 1.1.6 goes back to the 1920s to Suschkewitsch. The reader can find more about this in Lyubich's book [80].

The following theorem describes the structure of the space $X_{fl}(\mathcal{T})$ of flight vectors for a one-parameter weakly almost periodic semigroup (see [67]).

Theorem 1.1.14. *Let $\mathcal{T} = (T_t)_{t\in J}$ be a one-parameter weakly almost periodic semigroup in a Banach space X and let $x \in X$. Then the following conditions are equivalent*:

(i) $x \in X_{fl}(\mathcal{T})$;

(ii) *there exists a subsequence $(t_n)_{n=1}^\infty$ of J such that $T_{t_n}x \to 0$ weakly.*
Moreover, if $\mathcal{T}$ is a group then these conditions are equivalent to:

(iii) $\langle x, y\rangle = 0$ *for all eigenvectors of the adjoint semigroup $\mathcal{T}^*$ having unimodular eigenvalues.* □

1.1.11 To illustrate the power of Jacobs–Deleeuw–Glicksberg's theorem, we prove one well-known, but quite non-trivial, result.

Theorem 1.1.15. *Let T be a doubly power bounded positive operator in a finite-dimensional Banach lattice E. Then the operator T^{-1} is positive.*

Proof. The operator T is almost periodic since

$$\dim(E) < \infty \;\; \textit{and} \;\; \sup_{n\in\mathbb{N}} \|T^n\| < \infty\,.$$

By Jacobs–Deleeuw–Glicksberg's theorem,

$$E = E_r(T) \oplus E_{fl}(T)$$

and

$$E_{fl}(T) = \{x \in X : \lim_{n\to\infty} \|T^n x\| = 0\}.$$

The last condition together with the doubly power boundedness of T imply

$$E_{fl}(T) = \{0\},$$

and hence $E = E_r$. Then T^{-1} belongs to the closure (in the wo-topology) of the set $\{T^n\}_{n=1}^{\infty}$ of positive operators. Then T^{-1} is positive as well. □

It is interesting that this proposition cannot be generalized to the infinite-dimensional case [29] (see also the end of Section 2.2).

1.1.12 Another interesting application of the Jacobs–Deleeuw–Glicksberg theorem is the following result which is due to Lyubich and Phòng [83].

Theorem 1.1.16. *Let T be a power bounded operator in a Banach space X. Then the following conditions are equivalent*:

(i) $\lim_{n\to\infty} \|T^n x\| = 0$ *for all* $x \in X$;

(ii) T *is almost periodic and* $\lim_{n\to\infty} \|\mathcal{A}_n^{(\lambda^{-1}\cdot T)} x\| = 0$ *for all* $x \in X$ *and* $\lambda \in \sigma(T)$, $|\lambda| = 1$.

Proof. The implication (i) ⇒ (ii) is trivial. For proving (ii) ⇒ (i), we apply Theorem 1.1.4 to the almost periodic operator T, and get the decomposition

$$X = X_0(T) \oplus X_r(T),$$

where

$$X_0(\mathcal{T}) = \{x \in X : \lim_{t\to\infty} \|T_t x\| = 0\}$$

according to (1.1), and $X_r(T)$ is the closure of the linear span of all eigenvectors of T by (1.1). The condition (ii) says that there are no eigenvectors of T correspondent to eigenvalues λ, $|\lambda| = 1$, consequently, $X = X_0$. In other words, $\lim_{n\to\infty} \|T^n x\| = 0$ for all $x \in X$. □

1.1.13 Let X and Y be Banach spaces and let $R : X \to Y$ be a bounded linear operator. For each $T \in \mathcal{L}(X)$, we define the operator $\pi_R(T) \in \mathcal{L}(X \times Y)$ by

$$\pi_R(T)(x, y) := (Tx, (R - R \circ T)x + y) \qquad \Big((x, y) \in X \times Y\Big).$$

Proposition 1.1.17. *The mapping $\pi_R : \mathcal{L}(X) \to \mathcal{L}(X \times Y)$ (always nonlinear whenever $R \neq 0$) possesses the following properties*:

(i) π_R *is injective and* $\pi_R(S \circ T) = \pi_R(S) \circ \pi_R(T)$ *for all* $S, T \in \mathcal{L}(X)$;

(ii) $\pi_R(\alpha S + \beta T) = \alpha\pi_R(S) + \beta\pi_R(T)$ *for any* $S, T \in \mathcal{L}(X)$ *and* $\alpha, \beta \in \mathbb{C}$, $\alpha + \beta = 1$.

In particular, $(\pi_R(T_t))_{t\in J}$ *is a bounded one-parameter semigroup or* C_0*-semigroup, whenever* $(T_t)_{t\in J}$ *is.*

Proof. (i) The mapping π_R is, obviously, injective. Let $S, T \in \mathcal{L}(X)$, then for any $\tilde{x} := (x, y) \in X \times Y$ we obtain directly:

$$\begin{aligned}
\pi_R(S) \circ \pi_R(T)\tilde{x} &= \pi_R(S)\Big(Tx, (R - R\circ T)x + y\Big) \\
&= \Big(S \circ T(x), (R - R \circ S)Tx + [(R - R \circ T)x + y]\Big) \\
&= \Big(S \circ T(x), (R - R \circ S \circ T)x + y\Big) \\
&= \pi_R(S \circ T)\tilde{x}.
\end{aligned}$$

(ii) Given additionally $\alpha, \beta \in \mathbb{C}$, $\alpha + \beta = 1$, then

$$\begin{aligned}
\pi_R(\alpha S + \beta T)\tilde{x} &= \Big((\alpha S + \beta T)x, ((\alpha+\beta)R - R \circ (\alpha S + \beta T))x + (\alpha + \beta)y\Big) \\
&= \Big(\alpha Sx, \alpha(R - R \circ S)x + \alpha y\Big) + \Big(\beta Tx, \beta(R - R \circ T)x + \beta y\Big) \\
&= (\alpha\pi_R(S) + \beta\pi_R(T))\tilde{x}.
\end{aligned}$$

□

Proposition 1.1.17 is related to conservation laws. The *dilation* of semigroups given by π_R is useful in the investigation of their asymptotic behavior since it preserves asymptotic properties. For example, the following property can be obtained directly from Proposition 1.1.17.

Proposition 1.1.18. *A semigroup* $\mathcal{T}$ *is (weakly) almost periodic, strongly stable, asymptotically periodic, periodic, or mean ergodic if and only if the semigroup* $\pi_R(\mathcal{T})$ *possesses the same property.* □

1.1.14 There are some relations between weak almost periodicity and mean ergodicity. For instance, the following two assertions are direct corollaries of Theorem 1.1.7.

Proposition 1.1.19. *Any weakly almost periodic one-parameter semigroup* $\mathcal{T}$ *is mean ergodic.* □

Theorem 1.1.20. *Any bounded one-parameter semigroup* $\mathcal{T}$ *in a reflexive Banach space is mean ergodic.* □

1.1.15 An operator $T \in \mathcal{L}(X)$ is called an *isometry* if $\|Tx\| = \|x\|$ for all $x \in X$. It is easy to see that any doubly power bounded $T \in \mathcal{L}(X)$ becomes an invertible isometry with respect to the norm

$$\|x\|_T = \sup_{k\in\mathbb{Z}} \|T^k x\|.$$

As usual, denote by $\sigma(T)$ the *spectrum* of $T \in \mathcal{L}(X)$, and by

$$r(T) = \sup\{|\lambda| : \lambda \in \sigma(T)\} = \lim_{n\to\infty} \|T^n\|^{1/n}$$

its *spectral radius*. We shall use the notation

$$\sigma_\pi(T) := \{\lambda \in \sigma(T) : |\lambda| = r(T)\}$$

for the *peripheral spectrum* of T. The next result is due to Gelfand. We follow [6] in its proof.

Theorem 1.1.21 (Gelfand's lemma). *Let $T \in \mathcal{L}(X)$ be doubly power bounded with $\sigma(T) = \{1\}$, then $T = I$.*

Proof. Denote $M = \sup_{n\in\mathbb{Z}} \|T^n\| < \infty$. Let

$$S = \ln(T) = \sum_{k=1}^{\infty} \frac{(I-T)^k}{k},$$

then $\sigma(S) = \{0\}$ and $T = \exp(S)$. Thus, given any integer m, we have

$$\sigma(\sin(m\,S)) = \sin(\sigma(mS)) = \{0\}$$

and

$$\|(\sin(m\,S))^k\| = \left\| \left(\frac{T^m - T^{-m}}{2\,i} \right)^k \right\| \le M \qquad (\forall k \ge 0).$$

If $\sum_{k=0}^{\infty} c_k\, z^k$ is the Taylor expansion of the principal value of $\arcsin(z)$ about $z = 0$, then $c_k \ge 0$ for all k and

$$\sum_{k=0}^{\infty} c_k = \arcsin(1) = \pi/2\,.$$

Hence

$$\begin{aligned} \|m\,S\| &= \|\arcsin(\sin(m\,S))\| \\ &\le \sum_{k=0}^{\infty} |c_k| \cdot \|(\sin(m\,S))^k\| \\ &\le (\pi/2)\cdot M. \end{aligned}$$

As this holds for any integer m, it follows that $S = 0$. So, $T = I$. □

The following important generalization of Gelfand's lemma was obtained by Esterle [42], and by Katznelson and Tzafriri [61]. We use Phòng's elegant idea [95] in its proof.

Theorem 1.1.22 (Esterle–Katznelson–Tzafriri). *Let T be a power bounded operator in a Banach space X. Then*

$$\lim_{n\to\infty} \|T^{n+1} - T^n\| = 0$$

if and only if the peripheral spectrum $\sigma_\pi(T)$ of T satisfies $\sigma_\pi(T) \subseteq \{1\}$.

Proof. The "only if" part of this theorem is an easy exercise for the reader. We prove the "if" part. First we remark that it is enough to prove the following formerly weaker statement:

$$\sigma_\pi(T) \subseteq \{1\} \Rightarrow \lim_{n\to\infty} \|T^{n+1}x - T^n x\| = 0 \quad (\forall x \in X). \tag{1.8}$$

Indeed, suppose that (1.8) is proved. Let T be a power bounded operator in X such that $\sigma_\pi(T) \subseteq \{1\}$. Consider the operator $\overline{T}$ in $\mathcal{L}(X)$ defined by $\overline{T}(S) = T \circ S$. Then $\overline{T}$ is power-bounded and $\sigma(\overline{T}) = \sigma(T)$. Therefore,

$$\lim_{n\to\infty} \|\overline{T}^{n+1}(S) - \overline{T}^n(S)\| = 0 \quad (\forall S \in \mathcal{L}(X)).$$

Taking $S = I_X$, we get the desired conclusion.

Now, let T be a power bounded operator in X such that $\sigma_\pi(T) \subseteq \{1\}$. By using an equivalent norm in X, we can assume $\|T\| \leq 1$. Then the limit $l(x) = \lim_{n\to\infty} \|T^n x\|$ exists and it is a semi-norm in X. If $\ker(l) = X$, then the conclusion is obvious, and we may assume $\ker(l) \neq X$. It is clear that $\ker(l)$ is a closed invariant subspace of T, so T induces in a natural way an operator $\hat{T}$ on the quotient space $\hat{X} = X/\ker(l)$. The semi-norm l induces a norm $\hat{l}$ on $\hat{X}$ and, since T is isometric in the semi-norm l, the operator $\hat{T}$ is an isometry in the normed space $(\hat{X}, \hat{l})$. From the obvious inequality

$$l(R_\lambda(T)x) \leq \|R_\lambda(T)\| \cdot l(x)\,,$$

where $R_\lambda(T) = (\lambda I_X - T)^{-1}$ is the resolvent of T at $\lambda \notin \sigma(T)$, it follows that $\sigma(\hat{T}) \subseteq \sigma(T)$. Therefore $\hat{T}$ is an isometry with a single peripheral spectrum at 1. From this, it is easily seen that $\hat{T}$ is an invertible isometry with $\sigma(\hat{T}) = 1$. By Theorem 1.1.21, $\hat{T} = I_{\hat{X}}$. This means $\lim_{n\to\infty} \|T^n \circ (T - I)x\| = 0$ for all $x \in X$. The proof is completed. □

In contrast to Theorem 1.1.22, not any strongly stable T satisfies

$$\sigma_\pi(T) \subseteq \{1\}\,.$$

It is an easy exercise to construct $T \in \mathcal{L}(X)$ with $\lim_{n\to\infty} T^n x = 0$ for all x, while $\sigma_\pi(T) = \Gamma$. For various generalizations of the Gelfand lemma and Theorem 1.1.22, as well as for various results connected with them, we refer to the comprehensive survey of Zemanek [135]. We give here only one application of Theorem 1.1.22.

Proposition 1.1.23. *Let T be a power bounded mean ergodic operator such that $\sigma_\pi(T) \subseteq \{1\}$. Then T is strongly stable.*

Proof. By Theorem 1.1.9, $X = \mathrm{Fix}(T) \oplus \overline{N(T)}$. Thus the strong stability of T follows immediately from Theorem 1.1.22. □

We finish consideration of isometries with the following famous result of Godement [47].

Theorem 1.1.24 (Godement). *Let X be a complex Banach space and let T be an isometry in X. Suppose $\dim(X) > 1$, then T has a non-trivial invariant subspace.* □

Exercise 1.1.25. Let T be an isometry in a complex Banach space X, $\dim(X) = \infty$. Show that for any $n \in \mathbb{N}$ there is a family $\{X_k\}_{k=1}^n$ of T-invariant subspaces of X such that

$$\{0\} \subsetneqq X_1 \subsetneqq \cdots \subsetneqq X_n \subsetneqq X \,.$$

1.1.16 Let X be a complex Banach space. An operator semigroup $\mathcal{T} \subseteq \mathcal{L}(X)$ is called *supercyclic* if for some $x \in X$ (such an x is called a *supercyclic vector* for $\mathcal{T}$), the set

$$\{\lambda \cdot Tx : T \in \mathcal{T}, \lambda \in \mathbb{C}\}$$

is dense in X. An operator $T \in \mathcal{L}(X)$ is called *supercyclic* if the semigroup $(T^n)_{n=1}^\infty$ is supercyclic. The following interesting result about the asymptotic behavior of supercyclic operators was obtained in [10].

Theorem 1.1.26 (Ansari–Bourdon). *Let X be a complex Banach space such that $\dim(X) > 1$, and let $T \in \mathcal{L}(X)$ be a supercyclic power bounded operator. Then*

$$\lim_{n\to\infty} \|T^n x\| = 0$$

for all $x \in X$.

Proof. By passing to an equivalent norm, suppose $\|T\| \leq 1$. Assume $\|T^n x\| \not\to 0$ for some $x \in X$, $\|x\| = 1$. Without loss of generality, we may assume x to be supercyclic. Then $\|T^n x\| \downarrow \alpha > 0$. Since x is supercyclic, there exist an increasing sequence $(n_k)_{k=1}^\infty$ and $\lambda \in \mathbb{C}$ such that

$$\|T^n \circ T^{n_k} x - T^n(\lambda x)\| = \|T^n(T^{n_k} x - \lambda x)\| \to 0 \quad (k \to \infty) \tag{1.9}$$

for all $n \geq 0$. It follows from (1.9) that

$$\begin{aligned} |\lambda| = \|\lambda x\| &= \lim_{k\to\infty} \|T^{n_k} x\| = \alpha \\ &= \lim_{n\to\infty} \|T^n x\| \\ &= \lim_{n\to\infty} \lim_{k\to\infty} \|T^{n+n_k} x\| \\ &= \lim_{n\to\infty} \|T^n(\lambda x)\| \\ &= \lim_{k\to\infty} \|T^{n_k}(\lambda x)\| \\ &= |\lambda|^2, \end{aligned}$$

which implies $\alpha = |\lambda| = 1$. Using (1.9) again, we obtain

$$\lambda^m x \in \mathrm{cl}\,\{T^n x : n \geq 0\}$$

for all $m \in \mathbb{N}$ and, since $1 \in \mathrm{cl}\,\{\zeta^m : m \in \mathbb{N}\}$ for any $\zeta \in \mathbb{C}$, $|\zeta| = 1$, we have

$$\liminf_{n\to\infty} \|T^n x - x\| = 0\,. \tag{1.10}$$

Any vector satisfying (1.10) we call a *coming back vector* for T. Denote

$$Y = \mathrm{span}\{x, Tx, T^2x, \ldots, \}\,.$$

Take an arbitrary $y \in Y$:

$$y = \sum_{i=0}^{k} \alpha_i T^i x \quad (\alpha_0, \ldots, \alpha_k \in \mathbb{C})$$

and $\varepsilon > 0$. Since x is a coming back vector, there exists n_ε such that

$$\begin{aligned} \|T^{n_\varepsilon} y - y\| &\leq \sum_{i=0}^{k} |\alpha_i| \cdot \|T^i(T^{n_\varepsilon} x - x)\| \\ &\leq \|T^{n_\varepsilon} x - x\| \sum_{i=0}^{k} |\alpha_i| \\ &\leq \varepsilon\,. \end{aligned}$$

Thus Y consists of coming back vectors. Since $\|T\| \leq 1$, then every $y \in \mathrm{cl}\,(Y)$ is a coming back vector. It is obvious that $\mathrm{cl}\,(Y)$ is a T-invariant subspace of X. Applying supercyclicity again, we have $\mathrm{cl}\,(Y) = X$ and, therefore, every $z \in X$ is a coming back vector for T, which implies that T is an isometry.

Since $\dim(X) > 1$, Theorem 1.1.24 provides existence of a non-trivial T-invariant subspace which contradicts the supercyclicity of T. This shows that

$$\lim_{n\to\infty} \|T^n x\| = 0$$

for all $x \in X$. □

In Section 1.3, we give another approach to Theorem 1.1.26 based on the deep result of Storozhuk [124] (see Theorem 1.3.29).

1.1.17 Every power bounded operator is, obviously, Cesàro bounded and satisfies (1.7). However, there exist operators which are Cesàro bounded but do not satisfy (1.7). The following example of such an operator on $\mathbb{R}^2$ given by a matrix is due to Assani:

$$T = \begin{bmatrix} -1 & 2 \\ 0 & -1 \end{bmatrix}.$$

Exercise 1.1.27. Prove the above assertion about the operator T.

Exercise 1.1.28. Prove Theorem 1.1.7 in the strongly continuous case.

Exercise 1.1.29. Show that in the proof of Theorem 1.1.12 the use of the Eberlein theorem is not necessary.

1.1.18 The following result is due to Kakutani.

Theorem 1.1.30 (Kakutani). *The identity operator is an extreme point of the convex set of all contractions in any Banach space.*

Proof. Denote the set of all contractions in a Banach space X by C_X. If I_X is not an extreme point of C_X, then

$$I_X = 2^{-1}T_1 + 2^{-1}T_2$$

for some $T_1 \neq T_2$ in C_X. Denote $T = T_1 - I_X$.

If $h \in X^*$ is an extreme point of the closed unit ball B_{X^*} of X^*, then the identity

$$h = h \circ I_X = 2^{-1}h \circ T_1 + 2^{-1}h \circ T_2$$

shows $h = h \circ T_1 = h \circ T_2$. Then $T^*(h) = 0$ holds for all extreme point of B_{X^*}, and hence, by the Krein–Milman theorem, $T^*(x') = 0$ for all $x' \in B_{X^*}$. This contradicts our assumption that $T \neq 0$. □

Exercise 1.1.31. Let $(\alpha_n)_{n=1}^{\infty}$ be a sequence of strictly positive reals satisfying $\sum_{n=1}^{\infty} \alpha_n = 1$, and let $(T_n)_{n=1}^{\infty}$ be a sequence of pairwise-commuting power bounded operators in a Banach space X. Show that

$$\mathrm{Fix}(T) = \bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n),$$

where $T = \sum_{n=1}^{\infty} \alpha_n T_n$.

Hint: By using an appropriate equivalent norm, reduce this assertion to contractions and apply the previous proposition.

Exercise 1.1.32. Show that a convex combination of finitely many commuting power bounded mean ergodic operators in a Banach space is mean ergodic.

Hint: Use the previous exercise and Theorem 1.1.11.

1.1.19 We say that an operator $T \in \mathcal{L}(X)$ is *uniformly mean ergodic* if the sequence $(\mathcal{A}_n^T)_{n=1}^\infty$ converges in the operator norm. The following result of Lin [76] is the main theorem about uniform mean ergodicity. We follow Krengel [67, Thm.2.2.1] in its proof.

Theorem 1.1.33 (Lin). *Let an operator $T \in \mathcal{L}(X)$ satisfy $\lim_{n\to\infty} \|n^{-1}T^n\| = 0$. Then T is uniformly mean ergodic iff $(I-T)(X)$ is a closed subspace of X.*

Proof. Let T be uniformly ergodic, then $\overline{N(T)}$ is invariant under T, and the restriction S of T to $\overline{N(T)}$ satisfies $\|\mathcal{A}_n^S\| \to 0$. For any n with $\|\mathcal{A}_n^S\| < 1$, $I - \mathcal{A}_n^S$ is invertible. The invertibility of $I - S$, therefore, follows from the identity

$$(I-S)\left(\frac{n-1}{n}I + \frac{n-2}{n}S + \cdots + \frac{1}{n}S^{n-2}\right) = I - \mathcal{A}_n^S.$$

Hence

$$\begin{aligned} \overline{N(T)} &= (I-S)(\overline{N(T)}) \\ &= (I-T)(\overline{N(T)}) \\ &\subseteq (I-T)X \\ &= N(T)\,. \end{aligned}$$

Conversely, if $N(T) = \overline{N(T)}$, the open mapping theorem asserts that $(I-T)U$ is open in $\overline{N(T)}$ for any open $U \subseteq X$. Hence there exists $K > 0$ such that, for any $y \in \overline{N(T)}$, there is $z \in X$ with $(I-T)z = y$ and $\|z\| \le K\|y\|$ (otherwise, there is a sequence $(y_n)_{n=1}^\infty$ converging to 0 and disjoint to $(I-T)\{z : \|z\| < 1\}$). From

$$\begin{aligned} \|\mathcal{A}_n^T y\| &= \|\mathcal{A}_n^T (I-T)z\| \\ &\le n^{-1}\|I - T^n\|\|z\| \\ &\le Kn^{-1}\|I - T^n\|\|y\|, \end{aligned}$$

we see that the restriction S of T to $\overline{N(T)}$ is uniformly ergodic. It follows as above that $I - S$ is invertible on $\overline{N(T)}$, and

$$\begin{aligned} (I-T)X &= \overline{N(T)} \\ &= (I-S)(\overline{N(T)}) \\ &= (I-T)(\overline{N(T)}). \end{aligned}$$

Therefore, for any $x \in X$, there exists an element $y \in \overline{N(T)}$ with

$$(I-T)x = (I-T)y$$

and, by the invertibility of $(I-S)$, we may assume $\|y\| \le K'\|(I-T)x\|$ with some $K' < \infty$ independent on x. We now write

$$x = (x-y) + y\,.$$

As $(x-y)$ is fixed under T, we find

$$\begin{aligned}\|\mathcal{A}_n^T x - (x-y)\| &= \|\mathcal{A}_n^T y\| \\ &\le n^{-1}KK'\|I-T^n\|\|I-T\|\|x\|,\end{aligned}$$

i.e., the convergence of $\mathcal{A}_n^T$ to the projection P with $Px = x - y$ is uniform. □

There is an interesting spectral characterization of the uniform mean ergodicity; for the proof of it we send the reader to Krengel [67].

Theorem 1.1.34. *Let T be a power bounded operator in a Banach space X, then the following conditions are equivalent*:

(i) *T is uniformly mean ergodic*;

(ii) *either $I_X - T$ is invertible or* 1 *is a pole of first order of the resolvent $R_\theta(T)$ of T.*

Moreover, under these conditions, there are only finitely many poles of $R_\theta(T)$ in unit circle Γ of $\mathbb{C}$ and there no other singularities in $\sigma(T)\cap\Gamma$. □

Every mean ergodic operator in a finite-dimensional Banach space is, obviously, uniformly mean ergodic.

The dilation of semigroups given in **1.1.13** preserves the uniform ergodicity. The proof of this fact can be obtained directly from Proposition 1.1.17, and we leave it as an exercise. Another exercise follows.

Exercise 1.1.35. Show that if T^m is uniformly mean ergodic for some m, then T is uniformly mean ergodic.

Hint: Use the formula in the proof of Theorem 1.1.12.

The following definition is closely related to the uniform mean ergodicity.

Definition 1.1.36. A one-parameter semigroup $\mathcal{T}$ in a Banach space X is called *uniformly stable* if there exists a projection $P \in \mathcal{L}(X)$ such that $\lim\limits_{t\to\infty}\|T_t - P\| = 0$. An operator $T \in \mathcal{L}(X)$ is called *uniformly stable* if the semigroup $(T^n)_{n=1}^\infty$ is uniformly stable.

Exercise 1.1.37. Show that an operator $T \in \mathcal{L}(X)$ is uniformly stable iff T is uniformly mean ergodic and $\sigma_\pi(T) \subseteq \{1\}$.

Hint: See [99, *Lemma* 2.5].

1.1.20 An important class of uniformly mean ergodic operators which was introduced by Kryloff and Bogoliouboff is the class of so-called quasi-compact operators. An operator $T \in \mathcal{L}(X)$ is called *quasi-compact* if there is a sequence $(Q_n)_{n=1}^\infty$ of compact operators satisfying $\lim\limits_{n\to\infty}\|T^n - Q_n\| = 0$.

Exercise 1.1.38. Show that $T \in \mathcal{L}(X)$ is quasi-compact iff there exists a sequence $(T_n)_{n=1}^\infty$ of compact operators such that $\lim\limits_{n\to\infty}\|T^n - T_n\| = 0$.

Hint: See the proof of Lemma 2.4 in [67, *p*.88].

Exercise 1.1.39. Show that any power bounded quasi-compact operator is uniformly mean ergodic.

The following theorem gives a full description of power bounded quasi-compact operators. For its proof we send the reader to Krengel [67].

Theorem 1.1.40 (Yosida–Kakutani). *Let T be a power bounded quasi-compact operator in a Banach space X, then λT is quasi-compact and therefore uniformly mean ergodic for each $\lambda \in \Gamma$ (by Exercise 1.1.39) and, accordingly to Theorem 1.1.34, $\sigma(T) \cap \Gamma = \{\lambda_k\}_{k=1}^m$. Then*

$$T^n = \sum_{k=1}^{m} \lambda_k P_k + S^n \quad (\forall n \in \mathbb{N}),$$

where $P_k = \lim_{n\to\infty} \mathcal{A}_n^{\lambda_k^{-1}T}$ in the operator norm, and $S = T - \sum_{k=1}^m \lambda_k P_k$. Moreover, there exist constants η, M with $0 < \eta < 1$, $\|S^n\| \le M\eta^n$ for all $n \in \mathbb{N}$. □

1.1.21 We finish this section with a short discussion on complexifications. Recall that the *complexification* of a real vector space X is a complex vector space

$$X_{\mathbb{C}} := X \oplus iX = \{x + iy : \ x, y \in X\}$$

under the natural addition and under the following multiplication:

$$(a + ib) \cdot (x + iy) := ax - by + i(bx + ay).$$

We consider X as a real vector subspace of $X_{\mathbb{C}}$, by identifying $x \in X$ with $x + i0 \in X_{\mathbb{C}}$. Sometimes, an element $x + iy$ of $X_{\mathbb{C}}$ is denoted as a column vector

$$\begin{bmatrix} x \\ y \end{bmatrix}.$$

Obviously any complex vector space can be considered as a complexification of some real vector space. If X is a normed space, then $X_{\mathbb{C}}$ is also a normed space with the norm given by

$$\|x + iy\| := \sup_{\lambda\in[0,2\pi]} \|(\cos\lambda)x + (\sin\lambda)y\|. \tag{1.11}$$

Exercise 1.1.41. Show that $X_{\mathbb{C}}$ is a complex normed space under the norm given by (1.11). Show that $X_{\mathbb{C}}$ is a Banach space iff X is a Banach space.

Let $T : X \to X$ be a linear operator. Then its *complexification* $T_{\mathbb{C}} : X_{\mathbb{C}} \to X_{\mathbb{C}}$ is defined by

$$T_{\mathbb{C}}(x + iy) := Tx + iTy.$$

Obviously $T_{\mathbb{C}}$ is a linear operator in $X_{\mathbb{C}}$.

Exercise 1.1.42. Show that a vector subspace $Y \subseteq X$ is T-invariant iff the vector subspace $Y \oplus iY$ is $T_{\mathbb{C}}$-invariant.

Exercise 1.1.43. Let X be a normed space and let $T \in \mathcal{L}(X)$. Show that $T_{\mathbb{C}} \in \mathcal{L}(X_{\mathbb{C}})$ with $\|T_{\mathbb{C}}\| = \|T\|$. Show that T is invertible iff $T_{\mathbb{C}}$ is invertible and $T_{\mathbb{C}}^{-1} = (T^{-1})_{\mathbb{C}}$. Thus T is an isometry in X iff $T_{\mathbb{C}}$ is an isometry in $X_{\mathbb{C}}$.

Exercise 1.1.44. Let $Q : X_{\mathbb{C}} \to X_{\mathbb{C}}$ be a mapping. Show that Q is a linear operator in $X_{\mathbb{C}}$ iff there exist two uniquely determined linear operators $S, T : X \to X$ such that

$$Q\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} T & -S \\ S & T \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} Tx - Sy \\ Sx + Ty \end{bmatrix}.$$

1.2 Elementary theory of C_0-semigroups

Here we present the notion of a generator of a C_0-semigroup, which is a bridge between the abstract theory of C_0-semigroups and the important part of the theory of PDEs related to the so-called abstract Cauchy problem. We study elementary properties of generators and present one of the oldest classical results of the theory of C_0-semigroups, the Hille–Yosida theorem, that gives the conditions under which an (unbounded, in general) operator in a Banach space is a generator for some C_0-semigroup. We give its proof in the special case, for the C_0-semigroup of contractions. We discuss what would happen if we perturbed a generator of a C_0-semigroup by a bounded operator. We give (without a proof) the Dyson–Phillips formula for the perturbed semigroups. Then we touch the spectral theory of C_0-semigroups. We shall briefly discuss only a few results: the spectral mapping theorem and some extensions of the Liapunov stability theorem for ODEs on C_0-semigroups.

1.2.1 The starting point of the theory of C_0-semigroups is the *abstract Cauchy problem* (*ACP*):

$$\frac{du}{dt} = Au \quad (t \in \mathbb{R}_+), \quad u(0) = f, \tag{1.12}$$

where $u : \mathbb{R}_+ \to X$, X is a Banach space, A is a densely defined linear operator from $D(A) \subseteq X$ to X, and $f \in X$ is an *initial value*.

The ACP given by (1.12) is *well-posed* if a solution exists for all $f \in D(A)$, is unique, and depends continuously on A and f. We shall consider only well-posed ACPs.

Let $u(\,.\,, f)$ be a unique solution of the ACP (1.12) which corresponds to $f \in D(A)$. Thus, we have the linear mappings

$$T_t : D(A) \to X \quad (\forall t \geq 0)$$

defined by

$$T_t(f) := u(t, f) \in X \quad (\forall t \geq 0).$$

For any t, the mapping T_t is continuous and, therefore, possesses a unique continuous linear extension on X, which we denote also by T_t. Since

$$u(t+s,f) = u(t,u(s,f))$$

for all $t,s \geq 0$, we obtain the semigroup property of the family $\{T_t : t \in \mathbb{R}_+\}$ of operators:

$$T_t \circ T_s = T_{t+s} \quad (\forall t,s \geq 0).$$

For every $f \in D(A)$, the mapping $T_t f = u(t,f) : \mathbb{R}_+ \to X$ is differentiable and, therefore, continuous. The density of $D(A)$ and continuity of every T_t imply that the semigroup $(T_t)_{t\geq 0}$ is strongly continuous; and, since it satisfies

$$T_0 f = u(0,f) = f$$

for all $f \in D(A)$, $(T_t)_{t\geq 0}$ is a C_0-semigroup.

1.2.2 Let $\mathcal{T} = (T_t)_{t\geq 0}$ be a C_0-semigroup in a Banach space X.

Definition 1.2.1. The *generator* $G = G_{\mathcal{T}}$ of $\mathcal{T}$ is defined by

$$Gx = \|\cdot\| - \lim_{h\to 0} \frac{T_h x - x}{h}; \tag{1.13}$$

the *domain* of G, denoted as $D(G)$, consists of all $x \in X$ for which the limit (1.13) exists.

Obviously, $D(G)$ is a linear submanifold of X and G is a linear map from $D(G)$ to X. The set of all $\theta \in \mathbb{C}$ such that there exists a bounded operator $Q_\theta : X \to X$ satisfying

$$Q_\theta \circ (\theta\, I_X - G) = I_{D(G)} \quad \& \quad (\theta\, I_X - G) \circ Q_\theta = I_X$$

is called the *resolvent set* of G; the operator Q_θ is called the *resolvent* of G at the point θ, and is denoted by $R_\theta(G)$. The complement to the resolvent set is called the *spectrum* of G, and is denoted by $\sigma(G)$. According to **1.2.11**, the spectrum of a closed operator is always closed, but not necessarily a bounded subset of $\mathbb{C}$.

Remark that, by the definition of generator, if we consider the ACP with the generator G of a C_0-semigroup $\mathcal{T} = (T_t)_{t\geq 0}$ on the right-hand side, and present the solution of this ACP with use of a semigroup $\mathcal{S}$, then $\mathcal{S} = \mathcal{T}$. Thus any C_0-semigroup is related to some ACP.

1.2.3 Let us collect further important elementary properties of the generator of $\mathcal{T}$ in the following theorem.

Theorem 1.2.2. *Let G be the generator of $\mathcal{T}$.*

(i) *G commutes with T_t for all $t \geq 0$ in the sense that if $x \in X$ belongs to $D(G)$, so does $T_t x$, and*

$$G \circ T_t\, x = T_t \circ G\, x.$$

(ii) *The domain $D(G^n)$ of G^n is dense for any $n \in \mathbb{N}$.*

(iii) *The operator $G : D(G) \to X$ is closed.*

(iv) *Each $\theta \in \mathbb{C}$ such that $\mathrm{Re}(\theta) > \omega_{\mathcal{T}}$, where $\omega_{\mathcal{T}}$ is given by*

$$\omega_{\mathcal{T}} := \inf\{\alpha \in \mathbb{R} : \exists M_\alpha < \infty \text{ such that } \|T_t\| \leq M_\alpha e^{\alpha t} \text{ for all } t \geq 0\}, \tag{1.14}$$

belongs to the resolvent set of G. The resolvent of G is given by the Laplace transform *of $\mathcal{T}$:*

$$L_{\mathcal{T}}(\theta)x = \int_0^\infty e^{-\theta t} T_t\, x\, dt \quad (\forall x \in X). \tag{1.15}$$

Proof. We factor the difference quotient in (1.13) in two ways:

$$h^{-1}(T_{t+h} - T_t)\, x = h^{-1} T_t \circ (T_h - I_X)\, x = h^{-1}(T_h - I_X) \circ T_t\, x. \tag{1.16}$$

When $x \in D(G)$, the middle term converges to $T_t \circ G\, x$ as $h \to 0$. Therefore, the terms on the right and left converge also, and we deduce from (1.16) that

$$\frac{d}{dt}(T_t x) = T_t \circ G\, x = G \circ T_t\, x \quad (\forall x \in D(G)). \tag{1.17}$$

This proves (i).

(ii) We claim that an integrated form of (1.17)

$$T_t x - x = G \int_0^t T_\tau\, x\, d\tau \tag{1.18}$$

is valid for all x in X. Since $(T_t)_{t \geq 0}$ is strongly continuous, the integrand on the right in (1.18) is a continuous function of τ; therefore we may consider the integral as a vector-valued Riemann integral. To prove (1.18), we evaluate the action of G on this integral. Letting T_h act in the integrand and using the semigroup property, we get

$$\begin{aligned} h^{-1}(T_h - I_X) \int_0^t T_\tau\, x\, d\tau &= h^{-1} \int_0^t [T_{\tau+h}\, x - T_\tau\, x]\, d\tau \\ &= h^{-1} \int_t^{t+h} T_\tau\, x\, d\tau - h^{-1} \int_0^h T_\tau\, x\, d\tau \\ &\to T_t x - x, \end{aligned}$$

as $h \to 0$. Since the limit on the right exists and is equal to $T_t x - x$ for all $x \in X$, the limit on the left also exists and is equal to $G(\int_0^t T_\tau \, x \, d\tau)$. This proves (1.18), as well as $\int_0^t T_\tau \, x \, d\tau \in D(G)$ for all $x \in X$. Applying again

$$\lim_{t \to 0} t^{-1} \int_0^t T_\tau \, x \, d\tau = x \quad (\forall x \in X),$$

we obtain that $D(G)$ is dense in X.

We argue similarly about the domain of higher powers of G. Denote by ϕ an infinitely differentiable function on $\mathbb{R}$ supported on $[0, 1]$. For any x in X, we define

$$x_\phi = \int_0^{+\infty} \phi(\tau) T_\tau \, x \, d\tau.$$

The same argument as above shows that $x_\phi \in D(G)$ and

$$G x_\phi = -\int_0^{+\infty} \frac{d\phi}{d\tau}(\tau) T_\tau \, x \, d\tau.$$

Clearly, $x_\phi \in D(G^n)$ for all $n \in \mathbb{N}$. We choose now a sequence of ϕ_j supported on $[0, 1]$ that are non-negative, satisfy $\int_0^1 \phi_j \, d\tau = 1$, and whose support tends to zero. Appealing once more to strong continuity of $\mathcal{T}$, we conclude that x_{ϕ_j} tends to x. This proves that $D(G^n)$ is dense in X for all $n \in \mathbb{N}$.

(iii) We claim that the following integrated form of (1.17) is valid for all $x \in D(G)$:

$$T_t x - x = \int_0^t T_\tau \circ G \, x \, d\tau \, . \tag{1.19}$$

For proof, we appeal to the basic theorem of calculus for vector-valued functions: *if two functions whose value lies in a Banach space, and that have continuous strong derivatives, have the same derivative, and are equal at $t = 0$, then they are equal for all t.* We apply this theorem to the functions on two sides of (1.19). Both are 0 for $t = 0$. By (1.17), the derivative of the function on the left is $T_t \circ G \, x$. The derivative of the indefinite integral on the right is also $T_t \circ G \, x$. This proves (1.19).

To show that G is a closed operator, take a sequence $(x_n)_{n=1}^\infty$ in $D(G)$ such that $x_n \to x$, $G x_n \to y$. We claim that x lies in $D(G)$ and $Gx = y$. Take x to be x_n in (1.18),

$$T_t x_n - x_n = \int_0^t T_\tau \circ G \, x_n \, d\tau,$$

and let $n \to \infty$. Both sides converge, and their limits are equal:

$$T_t x - x = \int_0^t T_\tau \, y \, d\tau.$$

Divide by t, and let $t \to 0$. The right-hand side tends to y, the left-hand side tends to Gx. This shows that $x \in D(G)$ and $Gx = y$, which means that the operator G is closed.

(iv) The Laplace transform $L_{\mathcal{T}}(\theta)$ of $\mathcal{T}$ is defined according to (1.15), where the integral on the right is the limit of a vector-valued Riemann integral from 0 to N, as $N \to \infty$.

Since, by **1.2.13**, $\mathcal{T}$ grows at most exponentially, $\|T_t\| \leq M_k \exp(kt)$ for any $k > w_{\mathcal{T}}$, it follows that the integral on the right of (1.15) converges when $\mathrm{Re}\,(\theta) > k$ and that

$$\begin{aligned} \|L_{\mathcal{T}}(\theta)x\| &\leq \int_0^\infty M_k \cdot e^{(k-\mathrm{Re}(\theta))\tau} \cdot \|x\|\, d\tau \\ &= \frac{M_k}{\mathrm{Re}\,(\theta) - k}\|x\| \qquad (w_{\mathcal{T}} < k < \mathrm{Re}\,(\theta)). \end{aligned}$$

This shows that $L_{\mathcal{T}}(\theta)$ is a bounded operator and

$$\|L_{\mathcal{T}}(\theta)\| \leq \frac{M_k}{\mathrm{Re}\,(\theta) - k} \qquad (w_{\mathcal{T}} < k < \mathrm{Re}\,(\theta)). \tag{1.20}$$

We claim that $L_{\mathcal{T}}(\theta) = R_\theta(G)$, the resolvent of G at θ. To prove this, we look at the modified semigroup $(\exp(-\theta t)T_t)_{t\geq 0}$. It is easy to see that this is also a C_0-semigroup with the generator $G - \theta I_X$. We apply (1.18) to this modified semigroup:

$$\exp(-\theta t)T_t x - x = (G - \theta I_X) \int_0^t \exp(-\theta\tau)T_\tau x\, d\tau.$$

Suppose $\mathrm{Re}\,(\theta) > k > w_{\mathcal{T}}$; as $t \to \infty$, the left side tends to $-x$. The integral on the right tends to $L_{\mathcal{T}}(\theta)x$; since G is a closed operator, we conclude

$$x = (\theta I_X - G) \circ L_{\mathcal{T}}(\theta)\, x.$$

This shows that $L_{\mathcal{T}}(\theta)$ is a right inverse of $(\theta I_X - G)$. To deduce from this that $L_{\mathcal{T}}(\theta)$ is the inverse of $\theta I_X - G$, we use (1.19) instead of (1.18). □

1.2.4 Let X be a Banach space and let A be an operator in X. In the case when A is bounded, it is easy to construct a C_0-semigroup $\mathcal{T} = (T_t)_{t\geq 0}$ such that A becomes a generator of $\mathcal{T}$, namely:

$$T_t := \exp(tA) = \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n \qquad (t \in \mathbb{R}_+). \tag{1.21}$$

If the operator A is unbounded, the sum on the right-hand side of (1.21) does not exist. Another problem arises because A should satisfy some additional properties listed in Theorem 1.2.2 to be a generator of a C_0-semigroup.

We show how to reconstruct $\mathcal{T} = (T_t)_{t\geq 0}$ from its generator in the case when $\mathcal{T}$ consists of contractions, i.e., $\|T_t\| \leq 1$ for all $t \geq 0$.

Theorem 1.2.3 (Hille–Yosida). *The generator G of a C_0-semigroup $\mathcal{T}$ of contractions in a Banach space X has every real $\lambda > 0$ in its resolvent set, and*

$$\|R_\lambda(G)\| = \|(\lambda I_X - G)^{-1}\| \leq \lambda^{-1} \quad (\forall \lambda > 0). \tag{1.22}$$

Conversely, every densely defined operator G in X, whose resolvent set includes the positive reals and satisfies (1.22)*, is the generator of a C_0-semigroup of contractions.*

Proof. The first part is a restatement of (1.20).

We give Yosida's proof of the second part. It is based on approximating G by $G_n = n\, G \circ R_n(G)$ and letting $n \to \infty$. The identity

$$G_n = n^2 R_n(G) - n I_X \quad (n \in \mathbb{N}) \tag{1.23}$$

shows that G_n is a bounded operator. We approximate T_t by e^{tG_n}, where the exponential is defined as the infinite series in (1.21). We show first: if (1.22) holds, then, for all x in X,

$$\lim_{n\to\infty} n\, R_n(G)\, x = x. \tag{1.24}$$

We use the identity $n\, R_n(G) - I_X = R_n(G) \circ G$ and inequality (1.22) to deduce that

$$\begin{aligned} \|n\, R_n(G)\, x - x\| &= \|R_n(G) \circ G\, x\| \\ &\leq n^{-1} \|Gx\| \quad (\forall x \in D(G)). \end{aligned}$$

This proves (1.24) for $x \in D(G)$. Since, by (1.22), $\|n\, R_n(G)\| \leq 1$ for all $n \in \mathbb{N}$, and since $D(G)$ is dense in X, it follows that (1.24) holds for all $x \in X$.

Next we show that

$$\lim_{n\to\infty} G_n x = Gx \quad (\forall x \in D(G)). \tag{1.25}$$

By the definition of G_n,

$$G_n x = n\, G \circ R_n(G)\, x = n\, R_n(G) \circ G\, x \quad (\forall x \in D(G)),$$

so (1.25) follows from (1.24). Using formula (1.23), we obtain

$$e^{tG_n} = e^{-nt} e^{n^2 t\, R_n(G)} = e^{-nt} \sum_0^\infty \frac{(n^2 t)^m}{m!} R_n^m(G).$$

Using (1.22), we deduce that each e^{tG_n} is a contraction:

$$\|e^{tG_n}\| \leq e^{-nt} \sum \frac{(n^2 t)^m}{m!} \frac{1}{n^m} = e^{-nt} e^{nt} = 1. \tag{1.26}$$

To estimate the difference of e^{tG_n} and e^{tG_k}, we use the fact that e^{tG_k} and e^{tG_n} commute with G_n and G_k:

$$\frac{d}{dt}[e^{(s-t)G_n} \circ e^{tG_k}(x)] = e^{(s-t)G_n} \circ e^{tG_k} \circ (G_k - G_n)(x) \quad (\forall x \in X).$$

Using (1.26), we get

$$\|\frac{d}{dt}[e^{(s-t)G_n} \circ e^{tG_k}(x)]\| \leq \|G_k - G_n\|.$$

Integrating $\frac{d}{dt}[e^{(s-t)G_n} \circ e^{tG_k}(x)]$ with respect to t from 0 to s, we deduce from this inequality

$$\|e^{sG_n}x - e^{sG_k}x\| \leq s\,\|G_nx - G_kx\| \quad (\forall x \in X)\,. \tag{1.27}$$

Combining (1.25) and (1.27), we deduce that for all x in $D(G)$ the limit

$$\lim_{n\to\infty} e^{sG_n}\,x = T_sx \tag{1.28}$$

exists uniformly on every compact interval. It follows from the uniform boundedness of $(e^{tG_n})_{n=1}^{\infty}$ for any $t \geq 0$ that the limit in (1.28) exists for all $x \in X$. It follows directly that since $(e^{sG_n})_{s\geq 0}$ is a semigroup, so is $\mathcal{T} = (T_s)_{s\geq 0}$. Since the convergence in (1.28) is uniform on every compact interval $[0, t]$, strong continuity of the semigroup $(e^{sG_n})_{s\geq 0}$ implies strong continuity of $\mathcal{T}$. Since, by (1.26), each e^{sG_n} is a contraction, so is their strong limit $\mathcal{T}$.

It remains to show that the generator of $\mathcal{T}$ is G. Apply (1.20) to e^{tG_n}:

$$e^{tG_n}\,x - x = \int_0^t e^{sG_n} \circ\; G_n\,x\,ds\,.$$

Suppose that $x \in D(G)$ and let $n \to \infty$. So we get, using (1.25), that

$$T_tx - x = \int_0^t T_s \circ G\,x\,ds \quad (\forall x \in D(G)).$$

Denote by P the generator of $\mathcal{T}$. Dividing the above equation by t and letting $t \to 0$, we conclude that $D(P)$ includes $D(G)$ and $P = G$ on $D(G)$. Thus P is an extension of G. By Theorem 1.2.2, any $\lambda > 0$ belongs to the resolvent set of both G and P. Hence P cannot be a proper extension of G and so $P = G$. □

We give without proof the general case of Theorem 1.2.3 which is due to Feller, Miyadera, and Phillips.

Theorem 1.2.4. *Let G be an operator in a Banach space X, and let $w, M \in \mathbb{R}$, $M \geq 1$. Then the following conditions are equivalent*:

(i) *G is a generator of a C_0-semigroup $(T_t)_{t\geq 0}$ satisfying $\|T_t\| \leq M \exp(wt)$ for all $t \geq 0$;*

(ii) *G is a closed densely defined operator such that every $\theta > w$ belongs to the resolvent set of G and*

$$\|(\theta - w)^n\, R_\theta^n(G)\| \le M \quad (\forall n \in \mathbb{N});$$

(iii) *G is a closed densely defined operator such that every $\theta \in \mathbb{C}$ satisfying $\mathrm{Re}\,(\theta) > w$ belongs to the resolvent set of G and*

$$\|R_\theta^n(G)\| \le M \cdot |\mathrm{Re}\,(\theta) - w|^{-n} \quad (\forall n \in \mathbb{N}). \qquad \square$$

1.2.5 In many situations, the right-hand side of an ACP is given as a sum of several terms. Sometimes it is easy to find the solution of this ACP for each single term in the right-hand side, and the question arises how we can combine these solutions to get the solution of the initial ACP. On the language of C_0-semigroups, this question should be formulated as follows:

Question 1.2.5. Let G be the generator of a C_0-semigroup in a Banach space X and let F be an operator $X \supseteq D(F) :\to X$. Under what conditions is the sum $G + F$ the generator of some C_0-semigroup?

This question leads to the deep perturbation theory of C_0-semigroups. Here we mention without proof one result of this theory. For a bounded operator F, the answer to the question above is not very difficult and is given by the following theorem.

Theorem 1.2.6. *Let G be the generator of a C_0-semigroup $\mathcal{T} = (T_t)_{t\ge 0}$ in a Banach space X and $F \in \mathcal{L}(X)$. Then the operator $A = G + F : D(G) :\to X$ is the generator of another C_0-semigroup $\mathcal{S} = (S_t)_{t\ge 0}$ in X. Moreover*

$$S_t = T_t + \int_0^t T_{t-\tau} \circ F \circ S_\tau \, d\tau \quad (\forall t \ge 0), \tag{1.29}$$

where the integral is taken with respect to the strong operator topology on $\mathcal{L}(X)$. $\square$

The concrete form of this new C_0-semigroup is given by the following recurrent *Dyson–Phillips formula*:

$$S_t = \sum_{n=0}^{\infty} S_n(t), \quad \textit{where} \quad S_0(t) = T_t \quad \& \quad S_{n+1}(t) = \int_0^t T_{t-\tau} \circ F \circ S_n(\tau)\, d\tau\,.$$

Here, the infinite sum is taken in the operator norm on $\mathcal{L}(X)$, and the integral is defined with respect to the strong operator topology.

The proof of the following corollary is based on the Dyson–Phillips formula, on (1.29), and on Proposition 1.2.17 in **1.2.18**.

Corollary 1.2.7. *Let $(T_t)_{t\ge 0}$ be a C_0-semigroup in a Banach space X with the generator G and let K be a compact operator in X. Then $G + K$ is the generator of some C_0-semigroup $(S_t)_{t\ge 0}$ and the operator $T_t - S_t$ is compact for all $t \ge 0$.* $\square$

1.2.6 If the generator G of a C_0-semigroup $\mathcal{T} = (T_t)_{t\geq 0}$ is a bounded operator, $\mathcal{T}$ is the exponential of G:

$$T_t = e^{tG} = \sum_{n=0}^{\infty} \frac{t^n}{n!} G^n \quad (t \geq 0). \tag{1.30}$$

Indeed, if two C_0-semigroups $\mathcal{T}$ and $(e^{tG})_{t\geq 0}$ have the same generator G, then, according to **1.2.14**, they coincide. In this special case, by the spectral mapping theorem,

$$\sigma(T_t) = e^{t\sigma(G)} \quad (t \geq 0). \tag{1.31}$$

When G is unbounded, the representation (1.30) is no longer true. The question is: *does* (1.31) *hold*? In one direction, the answer is *yes*, and it is given by the following theorem which is due to Phillips.

Theorem 1.2.8 (Phillips). *Let $\mathcal{T} = (T_t)_{t\geq 0}$ be a C_0-semigroup with the generator G in a Banach space X, then*

$$e^{t\sigma(G)} \subseteq \sigma(T_t) \quad (\forall t \geq 0). \tag{1.32}$$

Proof. The operators T_t and $R_\zeta(G)$ commute for any $t \geq 0$ and ζ in the resolvent set of G. Adjoin to this family of operators their resolvents, and denote by $\mathcal{A}$ the closure in the uniform topology of the algebra generated by these operators. Then $\mathcal{A}$ is a commutative Banach algebra, and the spectrum of T_t and $R_\zeta(G)$ in $\mathcal{L}(X)$ is the same as their spectrum as elements of $\mathcal{A}$.

Let $\zeta > w_{\mathcal{T}}$ be fixed. We denote by $\{\mathcal{V}(t)\}_{t\geq 0}$ the following one-parameter family of operators

$$\mathcal{V}(t) = R_\zeta(G) \circ T_t. \tag{1.33}$$

The family $\mathcal{V}(t)$ depends continuously on t in the strong operator topology. Indeed, combine (1.15) with (1.33):

$$\begin{aligned} \mathcal{V}(t)x &= \int_0^\infty T_s\, e^{-\zeta s} T_t x\, ds \\ &= \int_0^\infty e^{-\zeta s} T_{s+t} x\, ds \\ &= e^{\zeta t} \int_t^\infty e^{-\zeta r} T_r x\, dr\,. \end{aligned}$$

From this, it is easy to see that $\mathcal{V}(t)$ depends continuously on t in the norm-topology on $\mathcal{L}(X)$. Since $\mathcal{T}$ commutes with $R_\zeta(G)$, it follows from (1.33) that

$$R_\zeta(G) \circ \mathcal{V}(t+s) = \mathcal{V}(t) \circ \mathcal{V}(s). \tag{1.34}$$

By **1.2.11**,

$$\sigma(R_\zeta(G)) = (\zeta - \sigma(G))^{-1}. \tag{1.35}$$

According to Gelfand's theory (see [105]),

$$\sigma(R_\zeta(G)) = \{h(R_\zeta(G)) : \text{ h is a homomorphism from } \mathcal{A} \text{ to } \mathbb{C}\}.$$

Combining this with (1.35), we conclude that for every $\gamma \in \sigma(G)$ there is $h \in \mathrm{Hom}_{\mathbb{C}}(\mathcal{A})$ such that

$$h(R_\zeta(G)) = (\zeta - \gamma)^{-1}. \tag{1.36}$$

Let h act on (1.34), then

$$h(R_\zeta(G) \circ \mathcal{V}(t+s)) = h(\mathcal{V}(t)) \cdot h(\mathcal{V}(s)). \tag{1.37}$$

It follows from (1.36) that $h(R_\zeta(G)) \neq 0$. We define

$$m(t) = \frac{h(\mathcal{V}(t))}{h(R_\zeta(G))} \tag{1.38}$$

and rewrite (1.37) as

$$m(t+s) = m(t) \cdot m(s). \tag{1.39}$$

We have shown above that $\mathcal{V}(t)$ is a continuous function of t in the uniform topology; the homomorphisms are continuous in the uniform topology. Combining these, we conclude that $h(\mathcal{V}(t))$ and, therefore, $m(t)$, are continuous functions from $\mathbb{R}$ to $\mathbb{C}$. It is well known that all non-trivial continuous solutions of (1.39) are of the form

$$m(t) = e^{kt} \quad (\forall t \geq 0). \tag{1.40}$$

Apply h to (1.33) to get

$$h(\mathcal{V}(t)) = h(R_\zeta(G)) \cdot h(T_t);$$

combining with (1.38) and (1.40) gives

$$h(T_t) = e^{kt} \quad (\forall t \geq 0). \tag{1.41}$$

Now multiply (1.15) by $R_\zeta(G)$:

$$R_\zeta(G)^2 x = \int_0^\infty e^{-\zeta s} R_\zeta(G) \circ T_s \, x \, ds.$$

As shown above, $R_\zeta(G) \circ T_s$ is continuous in the uniform topology; therefore, the integral above exists in the norm topology on $\mathcal{L}(X)$,

$$R_\zeta(G)^2 = \int_0^\infty e^{-\zeta s} R_\zeta(G) \circ T_s \, ds.$$

Apply h to both sides and use (1.41):

$$\begin{aligned} h(R_\zeta(G))^2 &= \int_0^\infty e^{-\zeta s} h(R_\zeta(G)) \cdot h(T_s)\, ds \\ &= h(R_\zeta(G)) \int_0^\infty e^{-\zeta s} e^{ks}\, ds \\ &= \frac{h(R_\zeta(G))}{\zeta - k}. \end{aligned}$$

Hence

$$h(R_\zeta(G)) = (\zeta - k)^{-1}.$$

Comparing with (1.36), we conclude that $k = \gamma$. Setting this into (1.41) gives

$$h(T_t) = e^{\gamma t}. \tag{1.42}$$

According to Gelfand's theory, $h(T_t) \in \sigma(T_t)$. Since γ is any point in $\sigma(G)$, we conclude from (1.42) that $\sigma(T_t)$ contains $e^{t\sigma(G)}$, as asserted in (1.32). □

1.2.7 The classical Liapunov stability theorem says that, for a well-posed ACP,

$$\frac{du}{dt} = Au \quad (t \in \mathbb{R}_+), \quad u(0) = f,$$

in a finite-dimensional Banach space X, the following conditions are equivalent:

(i) $\mathrm{Re}\,(\theta) < 0$ *for all* $\theta \in \sigma(A)$;

(ii) $\|u(t,f)\| \to 0$ *as* $t \to \infty$ *for the solution* u *of the ACP.*

Moreover, if these conditions hold, then $\|u(t,f)\| \to 0$ exponentially.

This theorem can be formulated in the language of C_0-semigroups in the following equivalent way.

Theorem 1.2.9 (Liapunov). *Let $(T_t)_{t\geq 0}$ be a C_0-semigroup in a finite-dimensional Banach space X with the generator A. Then the following conditions are equivalent*:

(i) $\mathrm{Re}\,(\theta) < 0$ *for all* $\theta \in \sigma(A)$;

(ii) $\lim_{t\to\infty} \|T_t x\| = 0$ *for all* $x \in X$.

The proof of this theorem is elementary and involves the Jordan decomposition of the matrix A. This theorem can be easily extended for uniformly continuous C_0-semigroups in any Banach space, if we replace the condition (ii) by

(ii)′ $\lim_{t\to\infty} \|T_t\| = 0$.

1.2.8 In general, the conditions (i) and (ii) are not equivalent, and the following principal question arises:

Under which additional conditions on a Banach space or on a C_0-semigroup we can provide the condition (ii)?

This question is of great importance in the analysis of asymptotic behavior of solutions of ACP. A very important contribution in the investigation of this question was made by Lyubich and Phòng [82], and independently, by Arendt and Batty [11]. We follow the Lyubich–Phòng approach to this result and its proof. We begin with the following lemma (see [82]).

Lemma 1.2.10. *Let $(U_t)_{t\geq 0}$ be a C_0-semigroup of isometries in a Banach space X with the generator S. If* $\mathrm{Re}\,(\lambda) < 0$*, then*

$$\|Sx - \lambda x\| \geq |\mathrm{Re}\,(\lambda)| \cdot \|x\| \tag{1.43}$$

for all $x \in D(S)$. Moreover, if $\sigma(S) \cap i\mathbb{R} \neq i\mathbb{R}$ then $\sigma(S) \subseteq i\mathbb{R}$.

Proof. Let $x \in D(S)$. We consider the X-valued function $u(t) = e^{-\lambda t} U_t\, x$, $t \geq 0$, then

$$\|u(t)\| = \exp\left(|\mathrm{Re}\,(\lambda)|\, t\right) \cdot \|x\|. \tag{1.44}$$

On the other hand,

$$u(t) = x + \int_0^t \frac{du(\tau)}{d\tau}\, d\tau = x + \int_0^t e^{-\lambda\tau} U_\tau (Sx - \lambda x)\, d\tau.$$

Therefore,

$$\|u(t)\| \leq \|x\| + \frac{\exp\left(|\mathrm{Re}\,(\lambda)|\, t\right) - 1}{|\mathrm{Re}\,(\lambda)|} \cdot \|Sx - \lambda x\|. \tag{1.45}$$

Comparing (1.44) and (1.45), we get (1.43).

Let $\sigma(S) \cap i\mathbb{R} \neq i\mathbb{R}$. For every $\lambda \in \mathbb{C}$, $\mathrm{Re}(\lambda) < 0$, denote

$$Q_\lambda = (\lambda I - S) : D(S) \to X\,.$$

The operator Q_λ has the closed range

$$Z_\lambda := Q_\lambda(D(S)) \subseteq X\,.$$

Indeed, if $y_n = Q_\lambda x_n \to y \in X$, then, by (1.43), $(x_n)_{n=1}^{\infty}$ is a Cauchy sequence and therefore $x_n \to x \in X$. Since Q_λ is a closed operator, x belongs to its domain and $y = Q_\lambda x \in Z_\lambda$.

Since $\sigma(S)$ is closed (by Exercise 1.2.14) and $\sigma(S) \cap i\mathbb{R} \neq i\mathbb{R}$, there exists

$$\lambda_0 \in \mathbb{C} \setminus \sigma(S)$$

such that $\mathrm{Re}(\lambda_0) < 0$. Thus the operator Q_{λ_0} is invertible and therefore $Z_{\lambda_0} = X$. If $Z_{\lambda_1} \neq Z_{\lambda_0} = X$ for some $\lambda_1 \neq \lambda_0$ in the left half-plane $\{\mathrm{Re}\lambda < 0\}$, then there is $y \in X = Z_{\lambda_0}$ such that $\|y\| = 1$ and $\mathrm{dist}(y, Z_{\lambda_1}) > 1/2$. Take $x \in X$ satisfying $y = Q_{\lambda_0}x = (\lambda_0 I - S)x$ and denote $u := Q_{\lambda_1}x = (\lambda_1 I - S)x \in Z_{\lambda_1}$. By (1.43),

$$\begin{aligned} 1/2 < \|y-u\| = |\lambda_0 - \lambda_1| \cdot \|x\| &\leq |\lambda_0 - \lambda_1| \cdot |\mathrm{Re}(\lambda_0)|^{-1} \cdot \|(\lambda_0 I - S)x\| \\ &= |\lambda_0 - \lambda_1| \cdot |\mathrm{Re}(\lambda_0)|^{-1}. \end{aligned}$$

This inequality shows that there exists an open neighborhood W of λ_0 such that $Z_{\lambda_1} = X$ for every $\lambda_1 \in W$. Hence $Z_\lambda = X$ for all λ in the left half-plane $\{\mathrm{Re}\lambda < 0\}$, and $\lambda I - S$ is invertible for all λ, $\mathrm{Re}\lambda < 0$. Since, by Theorem 1.2.2, $\lambda I - S$ is already invertible for all λ, $\mathrm{Re}\lambda > 0$, then $\sigma(S) \subseteq i\mathbb{R}$. □

Theorem 1.2.11 (Lyubich–Phòng). *Let G be the generator of a bounded C_0-semigroup $(T_t)_{t\geq 0}$ in a Banach space X. If the intersection of the spectrum of G with the imaginary axis is at most countable and the adjoint operator G^* has no imaginary eigenvalues, then a solution u_f of the ACP*

$$\frac{du}{dt} = Gu \quad (t \geq 0), \quad u(0) = f,$$

satisfies $\lim_{t\to\infty} \|u_f(t)\| = 0$ *for all* $f \in D(G)$.

Proof. We assume without loss of generality that the semigroup $(T_t)_{t\geq 0}$ consists of contractions. Then the function $\|T_t x\| : \mathbb{R}_+ \to \mathbb{R}_+$ is non-increasing for each fixed x, and hence the following limit exists:

$$l(x) = \lim_{t\to\infty} \|T_t x\| \qquad (x \in X).$$

The function $l : X \to \mathbb{R}_+$ is, obviously, a seminorm in X satisfying $l(x) \leq \|x\|$. We have to show that $l(x) \equiv 0$. For this, we consider the subspace $L = \ker(l)$ and suppose, on the contrary, that $L \neq X$. In the quotient space $\tilde{X} = X/L$, the seminorm l generates the norm $\tilde{l}$, and the semigroup $(T_t)_{t\geq 0}$ acts in $\tilde{X}$ in the natural way, because L is T_t-invariant for all $t \geq 0$. Since $l(T_s x) = l(x)$, $x \in X$, the corresponding operators $\overline{T_t} : \tilde{X} \to \tilde{X}$ are isometries. Moreover, the semigroup $(\overline{T_t})_{t\geq 0}$ is strongly continuous, because the seminorm l is dominated by the original norm in X. Denote by the same symbol $(\overline{T_t})_{t\geq 0}$ the unique extension of this C_0-semigroup onto the norm completion Y of $(\tilde{X}, \tilde{l})$. Let S be the generator of $(\overline{T_t})_{t\geq 0}$.

We show $\sigma(S) \subseteq \sigma(G)$. Let $\mu \notin \sigma(G)$. Since

$$l(R_\lambda(G)\, x) = \lim_{t\to\infty} \|R_\lambda(G) \circ T_t\, x\| \leq \|R_\lambda(G)\| \cdot l(x) \quad (\lambda \notin \sigma(G)),$$

the resolvent $R_\lambda(G)$ has a natural extension to a bounded operator $\overline{R_\lambda(G)}$ in Y. If $\mathrm{Re}(\lambda) > 0$, then, according to Theorem 1.2.2,

$$R_\lambda(G)\, x = \int_0^\infty e^{-\lambda t} T_t x\, dt \qquad (x \in X).$$

This implies

$$\overline{R_\lambda(G)}\,\hat{x} = \int_0^\infty e^{-\lambda t}\overline{T_t}\,\hat{x}\,dt \qquad (\hat{x} \in Y).$$

Therefore $\overline{R_\lambda(G)}$ coincides with the resolvent $R_\lambda(S)$ for all λ, $\mathrm{Re}(\lambda) > 0$. Now, by the resolvent identity

$$\overline{R_\mu(G)} - \overline{R_\lambda(G)} = (\lambda - \mu) \cdot \overline{R_\lambda(G)} \circ \overline{R_\mu(G)} \qquad (\forall \lambda \notin \sigma(G)),$$

we obtain

$$\overline{R_\mu(G)} - R_\lambda(S) = (\lambda - \mu) \cdot R_\lambda(S) \circ \overline{R_\mu(G)} \qquad (\mathrm{Re}(\lambda) > 0).$$

Therefore, $\mathrm{Im}(\overline{R_\mu(G)}) \subseteq D(S)$ and

$$(\lambda I_Y - S) \circ \overline{R_\mu(G)} = I_Y + (\lambda - \mu) \cdot \overline{R_\mu(G)} \quad (\mathrm{Re}(\lambda) > 0)\,.$$

From this, after excluding λ, we get

$$(\mu I_Y - S) \circ \overline{R_\mu(G)} = I_Y\,.$$

Analogously, we get

$$\overline{R_\mu(G)} \circ (\mu I_{D(S)} - S) = I_{D(S)}\,.$$

Thus $\mu \notin \sigma(S)$, and the resolvent $R_\mu(S)$ of S coincides with $\overline{R_\mu(G)}$ on $\mathbb{C} \setminus \sigma(G)$.

It follows from $\sigma(S) \subseteq \sigma(G)$ that the intersection $\sigma(S) \cap i\mathbb{R}$ is at most countable. Since S is the generator of C_0-semigroups of isometries satisfying

$$\sigma(S) \cap i\mathbb{R} \neq i\mathbb{R}\,,$$

it follows from Lemma 1.2.10 that $\sigma(S) \subseteq i\mathbb{R}$. Moreover, $\sigma(S) \neq \emptyset$ since S is the generator of a group of isometries. Thus $\sigma(S)$ is a non-empty, at most countable, closed subset of $i\mathbb{R}$. Therefore, it contains an isolated point $i\omega \in i\mathbb{R}$. The Riesz projection $P \neq 0$ corresponding to $\{i\omega\}$ commutes with S and with all $\overline{T_t}$.

The subspace $\Omega = \mathrm{Im}(P)$ is invariant for S and for all $\overline{T_t}$, and the corresponding C_0-semigroup of isometries $U_\omega(t) = \overline{T_t}|_\Omega$ in Ω is generated by the bounded operator $S_\omega = S|_\Omega$ with the spectrum $\sigma(S_\omega) = \{i\omega\}$, and therefore $(U_\omega(t))_{t\geq 0}$ is uniformly continuous.

The spectral mapping theorem for uniformly continuous semigroups (see Theorem 1.2.18) implies that $\sigma(U_\omega(t)) = e^{it\omega}$ for all $t \geq 0$. According to Theorem 1.1.21, $U_\omega(t) = e^{it\omega} I_\Omega$ for all $t \geq 0$, and hence $S_\omega = i\omega I_\Omega$. Therefore, Ω is an eigenspace of S corresponding to the eigenvalue $i\omega$. Then every linear functional $h \in \mathrm{Im}(P^*)$, $h \neq 0$, is an eigenfunctional for S^* with the same eigenvalue. We extend h to the whole of X by using the homomorphisms $X \to \tilde{X} \to Y$. We get a nonzero functional $f \in X^*$ which is an eigenfunctional for G^* with the eigenvalue $i\omega$; a contradiction. □

1.2.9 Theorem 1.2.11 is more known in the following equivalent form, in which it was stated independently by Arendt and Batty [11].

Theorem 1.2.12 (Arendt–Batty–Lyubich–Phòng). *Let G be the generator of a bounded C_0-semigroup $(T_t)_{t\geq 0}$ in a Banach space X. If $\sigma(G)\cap i\mathbb{R}$ is at most countable and the adjoint operator G^* has no imaginary eigenvalues, then $\lim_{t\to\infty} \|T_t x\| = 0$ for all $x \in X$.* □

Since in a reflexive Banach space the point spectrum $\sigma_p(G)$ of G coincides with $\sigma_p(G^*)$, the condition $\sigma_p(G^*) \cap i\mathbb{R} = \emptyset$ in the reflexive case can be replaced by the more simple condition $\sigma_p(G) \cap i\mathbb{R} = \emptyset$.

Another result of such type, which can be obtained from Theorem 1.2.12, is the following condition for the almost periodicity of a C_0-semigroup [83].

Theorem 1.2.13 (Lyubich–Phòng). *Let G be the generator of a bounded C_0-semigroup $\mathcal{T} = (T_t)_{t\geq 0}$ such that $\sigma(G)\cap i\mathbb{R}$ is at most countable, and the C_0-semigroup $(e^{-\lambda}T_t)_{t\geq 0}$ is mean ergodic for every $\lambda \in \sigma(G) \cap i\mathbb{R}$. Then the semigroup $\mathcal{T}$ is almost periodic.* □

Related Results and Notes

1.2.10 Let us give several examples of ACP. They come from the following PDE:

a) the *heat equation* $u_t = \Delta u\,;$

b) the *wave equation* $u_{tt} = \Delta u\,;$

c) the *Schrödinger equation* $u_t = -i\Delta u\,.$

The wave equation can be written as ACP if we reduce it to a first order system, namely set

$$\overline{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} u \\ u_t \end{bmatrix}, \quad A = \begin{bmatrix} 0 & Id \\ \Delta & 0 \end{bmatrix}.$$

Then the heat equation is equivalent to ACP: $\overline{v}_t = A\overline{v}$ on a space of pairs of functions.

1.2.11 Establish the following properties of an operator.

Exercise 1.2.14. Let $G : D(G) \to X$ be a linear operator in a Banach space X. Show that if $\sigma(G) \neq \mathbb{C}$, then G is closed. Show that the spectrum $\sigma(G)$ of any closed operator G is a closed set. Show that

$$\sigma(R_\zeta(G)) = (\zeta - \sigma(G))^{-1}$$

for all ζ in the resolvent set of G. Construct an example of a densely defined closed operator whose spectrum is empty.

1.2.12 The strong continuity of a semigroup $\mathcal{T} = (T_t)_{t\geq 0}$ in a Banach space X is equivalent to the strong continuity at 0 and equivalent to the weak continuity at 0. We prove only the first equivalence and refer the reader to a little bit technical proof of the second to Hille–Phillips [57], or Yosida [130]. Let $\lim_{t\to 0} T_t x = x$ for all $x \in X$.

First we show that $\mathcal{T}$ is uniformly bounded on every compact interval $[0, \alpha]$. Since $\mathcal{T}$ is a semigroup, it is enough to show that the set $\{\|T_t\| : 0 \leq t \leq \alpha\}$ is bounded for some $\alpha > 0$. Assume that it is not true. Then there exists a sequence $(t_n)_{n=1}^{\infty}$ of positive reals convergent to 0 such that $\|T_{t_n}\| \to \infty$. Then, by the uniform boundedness principle, $\sup_n \|T_{t_n} x\| = \infty$ for some $x \in X$, which contradicts the strong continuity of $\mathcal{T}$ at 0.

For a given $x \in X$, take a real $t > 0$. Then the right continuity at t follows from

$$\lim_{s\to 0+} \|T_{t+s}x - T_t x\| \leq \|T_t\| \cdot \lim_{s\to 0+} \|T_s x - x\| = 0.$$

If $-t \leq s \leq 0$, then the inequality

$$\|T_{t+s}x - T_t x\| \leq \|T_{t+s}\| \cdot \|T_{-s}x - x\|$$

shows the left continuity, since $\|T_t\|$ is bounded on $[0, t]$.

1.2.13 Now we show that the number $w_{\mathcal{T}}$ defined in (1.14) is finite for any C_0-semigroup $\mathcal{T}$. Take $\alpha > 0$ then, as it was shown in the previous subsection, $\mathcal{T}$ is bounded in the norm on the interval $[0, \alpha]$ by some constant, say β. Any $t \geq 0$ can be decomposed as $t = n\alpha + \vartheta$, $0 \leq \vartheta < \alpha$. Then $T_t = T_\alpha^n \circ \mathcal{T}_\vartheta$, and

$$\|T_t\| \leq \|T_\alpha\|^n \|\mathcal{T}_\vartheta\| \leq \beta^{n+1} \leq \beta \exp(kt),$$

where $k = \alpha^{-1} \ln \beta$. Thus $w_{\mathcal{T}} \leq k < \infty$.

1.2.14 Show that the generator G of a C_0-semigroup $\mathcal{T}$ determines $\mathcal{T}$ uniquely.

Hint: Let G be the generator of another C_0-semigroup $\mathcal{S}$. Fix $q > 0$, and consider the map

$$t \to \Psi_x(t) := T_{q-t} \circ S_t\,(x) \qquad (0 \leq t \leq q\,;\, x \in D(G)).$$

Prove that $\frac{d\Psi_x}{dt}(t) = 0$ for all $0 \leq t \leq q$ and $x \in D(G)$. Then use $\Psi_x(0) = T_q x$, $\Psi_x(q) = S_q x$, and the density of $D(G)$.

Show that if $\mathcal{T} = (T_t)_{t\geq 0}$ is a C_0-semigroup of isometries and $\sigma(G) \subseteq i\mathbb{R}$, where G is the generator of $\mathcal{T}$, then $\mathcal{T}$ can be extended to the group of isometries and $\sigma(G) \neq \emptyset$.

Hint: Use Theorem 1.2.3

1.2.15 Another type of continuity of a C_0-semigroup $\mathcal{T} = (T_t)_{t\geq 0}$ is the *uniform continuity*, i.e., the continuity with respect to the operator norm:

$$\lim_{s\to t} \|T_s - T_t\| = 0 \quad (\forall t \geq 0).$$

Uniform continuity of $\mathcal{T}$ is equivalent to uniform continuity at 0: $\lim_{s\to 0} \|T_s - I\| = 0$. The proof of this easy fact is left to the reader as an exercise. Another simple exercise is to prove the following proposition.

Proposition 1.2.15. *Let G be the generator of a C_0-semigroup $\mathcal{T}$ in a Banach space X. Then the following conditions are equivalent*:

(i) *G is a bounded operator in X*;

(ii) *$\mathcal{T}$ is uniformly continuous.* □

A C_0-semigroup $\mathcal{T} = (T_t)_{t\geq 0}$ is called *uniformly mean ergodic* if $\lim\limits_{n\to\infty} \mathcal{A}_t^{\mathcal{T}}$ exists in the operator norm. Let $\mathcal{T} = (T_t)_{t\geq 0}$ be a C_0-semigroup with the generator G. Show that $\mathcal{T}$ is uniformly mean ergodic iff $0 \notin \sigma(G)$ or 0 is a first-order pole of $R_\lambda(G)$.The following characterization of uniformly mean ergodic C_0-semigroups, which is similar to Theorem 1.1.33, is due to Shaw [115, Thm.4].

Theorem 1.2.16 (Shaw). *Let $\mathcal{T} = (T_t)_{t\geq 0}$ be a C_0-semigroup in X with the generator A such that the Laplace transform $L_{\mathcal{T}}(\theta)x$ exists for all $x \in X$ and for all $\theta \in \mathbb{C}$, $\mathrm{Re}(\theta) > 0$. Then the following conditions are equivalent*:

(i) *$\mathcal{T}$ is uniformly mean ergodic*;

(ii) *$\lim\limits_{t\to\infty} t^{-1}\|T_t \circ R_1(A)\| = 0$ and $(I_X - R_1(A))$ is closed.* □

1.2.16 One more proof of Theorem 1.2.3, which is due to Hille, is based on the approximation by

$$W_n(t) = \Big(\frac{n}{t} R_{\frac{n}{t}}(G)\Big)^n \quad (n \in \mathbb{N},\, t \geq 0).$$

We leave the verification of the following properties to the reader:

(i) each $W_n(t)$ is a contraction;

(ii) $W_n(t)$ converges strongly to a semigroup whose generator is G.

1.2.17 Let X be an L^1-space and let G be a densely defined operator $D(X) \to X$ whose resolvent $R_\lambda(G)$ exists for all $\lambda > 0$ and satisfies (1.22).

Show that if $\lambda R_\lambda(G)$ is a Markov operator on X for all $\lambda > 0$, then the C_0-semigroup generated by G according to the Hille–Yosida theorem consists of Markov operators (see the definition of Markov operator in L^1-space in **3.1.3**). Is it possible to replace the condition that $\lambda R_\lambda(G)$ is a Markov operator for all $\lambda > 0$ by the formally weaker one that $\lambda R_\lambda(G)$ is a Markov operator for some $\lambda > 0$?

1.2.18 Prove the following property of the vector-valued Riemann integral with respect to the strong operator topology.

Proposition 1.2.17. *If a function $\phi : [0,1] \to \mathcal{L}(X)$ is continuous with respect to the strong operator topology on $\mathcal{L}(X)$ and $\phi(t)$ is compact for all $t \in [0,1]$, then the operator $\int_0^1 \phi(t)\,dt$ is compact.* □

1.2.19 The inclusion in (1.32) is proper in some cases. However, in the uniformly continuous case, we have the equality.

Theorem 1.2.18 (Phillips). *Let $(T_t)_{t\geq 0}$ be a C_0-semigroup with the generator G. Suppose that there exists a $t_0 > 0$ such that T_t is uniformly continuous for all $t \geq t_0$. Then $\sigma(T_t) = e^{t\sigma(G)}$ for all $t \geq t_0$.*

Proof. In the notation of Theorem 1.2.8, the spectrum of T_t is the set $\{h(T_t) : h \in \mathrm{Hom}_{\mathbb{C}}(\mathcal{A})\}$. Applying h to $T_{s+t} = T_s \circ T_t$, we get $h(T_{s+t}) = h(T_s) \cdot h(T_t)$. Since T_t is assumed uniformly continuous for $t \geq t_0$, it follows from the definition that the map $t \to h(T_t)$ is continuous for $t \geq t_0$. The non-trivial solutions of the functional equation above are the exponentials $h(T_t) = e^{\nu t}$. The rest of the proof proceeds as that of Theorem 1.2.8. □

1.2.20 It is easy to show that if γ is an eigenvalue of G, then $e^{t\gamma}$ is an eigenvalue of T_t for each $t \geq 0$. To see this, let u be a corresponding eigenvector: $G\,u = \gamma u$. Then

$$\frac{d}{dt} e^{-\gamma t} T_t\, u = e^{-\gamma t} T_t \circ (G - \gamma I_X)\, u = 0 \quad (t \geq 0),$$

which means that $e^{-\gamma t} T_t\, u = u$. Since $e^{-\gamma 0} T_0\, u = u$, then $e^{-\gamma t} T_t\, u = u$ identically. This shows that $e^{\gamma t}$ is an eigenvalue of T_t for all $t \geq 0$.

1.2.21 We consider the question of the transpose of a semigroup and of its generator.

Theorem 1.2.19. *Let X be a reflexive Banach space and let $(T_t)_{t\geq 0}$ be a C_0-semigroup in X. Then the transpose semigroup ${T^*}_{t\geq 0}$ in X is likewise a C_0-semigroup in X^*, whose generator is the transpose of the generator of $(T_t)_{t\geq 0}$.*

Proof. By definition of the transpose,

$$\langle T_t x, f\rangle = \langle x, {T_t}^* f\rangle \qquad (x \in X\,, f \in X^*).$$

From this and reflexivity of X, we deduce that $(T_t^*)_{t\geq 0}$ is weakly sequentially continuous. But then, by **1.2.12**, $(T_t^*)_{t\geq 0}$ is strongly continuous.

We leave the proof of the statement of the theorem about the generator of $(T_t^*)_{t\geq 0}$ to the reader. □

1.2.22 Prove the following theorem.

Theorem 1.2.20. *Let $(T_t)_{t\geq 0}$ be a uniformly continuous C_0-semigroup in a Banach space X with the generator G. Then the following conditions are equivalent*:

(a) $\mathrm{Re}\,(\theta) < 0$ *for all* $\theta \in \sigma(A)$;

(b) $\lim\limits_{t\to\infty} \|T_t\| = 0$. □

1.2.23 The following stability result of Phòng [96] can be derived from Theorem 1.2.12, but we prefer to give its direct proof according to [96].

Theorem 1.2.21 (Phòng). *Let $\mathcal{T} = (T_t)_{t\geq 0}$ be a bounded C_0-semigroup in a Banach space X with the generator G. Assume that there is $t_0 > 0$ such that T_{t_0} commutes with a compact operator with dense range. Then $\lim\limits_{t\to\infty} \|T_t x\| = 0$ for all $x \in X$ if and only if $\sigma_p(G) \cap i\mathbb{R} = \emptyset$.* □

Theorem 1.2.21 follows from the more general Theorem 1.2.22. First of all, we need a definition. Let $\mathcal{S} = (S_t)_{t\geq 0}$ be another bounded C_0-semigroup in a Banach space Y. We say that an operator $C : Y \to X$ *intertwines* $\mathcal{T}$ with $\mathcal{S}$ if $T_t \circ C = C \circ S_t$ for all $t \geq 0$.

Theorem 1.2.22 (Phòng). *Let $\mathcal{T} = (T_t)_{t\geq 0}$ be a bounded C_0-semigroup in a Banach space X with the generator G, and let $\mathcal{S} = (S_t)_{t\geq 0}$ be a bounded C_0-semigroup in a Banach space Y. Assume that there exists an operator $C : Y \to X$ with dense range, which intertwines $\mathcal{T}$ with $\mathcal{S}$, and assume that there is $t_0 > 0$ such that S_{t_0} commutes with a compact operator with dense range. Then $\lim\limits_{t\to\infty} \|T_t x\| = 0$ for all $x \in X$ if and only if $\sigma_p(G) \cap i\mathbb{R} = \emptyset$.*

Proof. The proof of the "only if" part is obvious. The proof of the "if" part is an application of Theorem 1.1.4, which implies that if $\mathcal{T}$ is almost periodic, then, for the condition $\lim\limits_{t\to\infty} \|T_t x\| = 0$ for all $x \in X$, it is enough to have $\sigma_p(G) \cap i\mathbb{R} = \emptyset$. Indeed, suppose that $\mathcal{T}$ is almost periodic, then

$$X_r(\mathcal{T}) = \overline{\mathrm{span}}\{x \in X : \exists \text{ a character } \alpha : \mathbb{R}_+ \to \Gamma \text{ s.t. } \forall t \in \mathbb{R}_+ \; T_t x = \alpha(t)x\}\,,$$

according to Theorem 1.1.4, (1.1). If $X_r(\mathcal{T}) \neq \{0\}$, then there exist $0 \neq x \in X$ and a character $\alpha : \mathbb{R}_+ \to \Gamma$ satisfying $T_t x = \alpha(t)x$ for all $t \geq 0$. Since $\mathcal{T}$ is a C_0-semigroup, $\alpha(t)$ is a continuous function satisfying the equality $\alpha(t+s) = \alpha(t)\cdot\alpha(s)$ for all $t, s \geq 0$. Thus $\alpha(t) = e^{kt}$ for an appropriate $k \in i\mathbb{R}$. Then k belongs to $\sigma_p(G) \cap i\mathbb{R}$ and corresponds to the eigenvector x, a contradiction. This remark reduces the proof of our theorem to the proof of almost periodicity of $\mathcal{T}$.

Now, since $T_t \circ C = C \circ S_t$ for all $t \geq 0$, and $S_{t_0} \circ K = K \circ S_{t_0}$ for some compact $K : Y \to Y$, it follows that $T_{t_0} \circ C \circ K = C \circ K \circ S_{t_0}$, and hence $T_{nt_0} \circ C \circ K = C \circ K \circ S_{nt_0}$ for all $n = 0, 1, 2, \ldots$, and hence $\{T_t\, x\}_{t\geq 0}$ is relatively compact for every $x \in C \circ K\,(Y)$. Since C and K have dense ranges, $C \circ K\,(Y)$ is dense in X, and since $\mathcal{T}$ is a bounded semigroup, it follows that $\{T_t\, x\}_{t\geq 0}$ is relatively compact for each x in X, i.e., $\mathcal{T}$ is almost periodic. □

Results analogous to Theorems 1.2.21 and 1.2.22 also hold for a discrete one-parameter semigroup generated by a single operator T.

Theorem 1.2.23 (Phòng). *Let T be a power bounded operator in a Banach space X and let S be a power bounded operator in a Banach space Y. Assume that: (i) there exists an operator $C : Y \to X$ with dense range such that $T \circ C = C \circ S$; and (ii) S commutes with some compact operator with dense range. Then $\lim_{n\to\infty} \|T^n x\| = 0$ for all $x \in X$ if and only if $\sigma_p(T) \cap \Gamma = \emptyset$, where Γ is the unit circle in $\mathbb{C}$. In particular, if T commutes with a compact operator with dense range and $\sigma_p(T) \cap \Gamma = \emptyset$, then $\lim_{n\to\infty} \|T^n x\| = 0$ for all $x \in X$.* □

1.2.24 There are discrete versions of Theorems 1.2.12 and 1.2.13, which are due to Lyubich and Phòng [83] (see also [73]).

Theorem 1.2.24 (Lyubich–Phòng). *Let T be a power bounded operator in a Banach space X such that $\sigma(T) \cap \Gamma$ is at most countable, where $\Gamma := \{\lambda \in \mathbb{C} : |\lambda| = 1\}$, and $\lambda^{-1} \cdot T$ is mean ergodic for every $\lambda \in \Gamma$. Then T is almost periodic.* □

Theorem 1.2.25 (Lyubich–Phòng). *Let T be a power bounded operator in a Banach space X such that $\sigma(T) \cap \Gamma$ is at most countable and let $\sigma(T^*) \cap \Gamma = \emptyset$. Then $\lim_{n\to\infty} \|T^n x\| = 0$ for all $x \in X$.* □

1.2.25 Let us mention only a few results concerning the so-called *individual behavior* of orbits of one-parameter semigroups. This area of research is quite recent and promises to be very productive. One of the oldest and, probably, most famous result here is the following theorem, which is due to Datko [22].

Theorem 1.2.26 (Datko). *A C_0-semigroup $\mathcal{T} = (T_t)_{t\geq 0}$ in a Banach space X is* uniformly exponentially stable *(i.e., the constant $\omega(\mathcal{T})$ which is defined in (1.14) strictly less then 1) if and only if all functions $T_t x : \mathbb{R}_+ \to X$ (for all $x \in X$) belong to $L^p(\mathbb{R}_+, X)$ for some fixed p, $1 \leq p < \infty$.* □

We mention also the result, which is due to Müller [86] and van Neerven [88], about the individual behavior of orbits of a single operator. Its analogue for C_0-semigroups is obvious.

Theorem 1.2.27 (Müller–van Neerven). *Let T be a power bounded operator with the spectral radius $r(T) = 1$ in a Banach space X. Then, for any $\varepsilon > 0$ and positive sequence $(\alpha_n)_{n=1}^{\infty}$ in c_0 with $\sup_n \alpha_n = 1$, there exists $x \in X$, $\|x\| = 1$, satisfying $\|T^k x\| \geq (1 - \varepsilon)\alpha_k$ for all k.* □

For other interesting results related to this topic, we refer to [90], and [123].

1.3 Constrictive and quasi-constrictive semigroups

This section is devoted to one-parameter operator semigroups in arbitrary Banach spaces, such that their asymptotic behavior is, in some reasonable sense, finite-dimensional. In the investigation of such semigroups, the methods of linear algebra can be applied in various ways. Such semigroups appear often in different applications and henceforth their study is well motivated. We give some examples and refer for others to the Lasota–Mackey book [71]. We begin with the notion of a constrictor of an operator semigroup, and then investigate semigroups possessing compact or quasi-compact constrictors.

1.3.1 Let X be a Banach space with a norm $\|\cdot\|$. We denote by B_X the closed unit ball of X, and by $\operatorname{dist}(y, A)$ the *distance* from $y \in X$ to a subset A of X. Let $\mathcal{T} = (T_t)_{t\in J}$ be a one-parameter operator semigroup in X.

Definition 1.3.1. We call a subset $A \subseteq X$ a *constrictor* for $\mathcal{T}$ if

$$\lim_{t\to\infty} \operatorname{dist}(T_t x, A) = 0$$

for each $x \in B_X$.

Of course, the notion of constrictor depends on the choice of a norm in X. We denote the set of all constrictors for $\mathcal{T}$ by $\operatorname{Constr}_{\|\cdot\|}(\mathcal{T})$ or, simply, $\operatorname{Constr}(\mathcal{T})$, if the norm in X is fixed.

In the discrete case, we say that A is a *constrictor* for an operator T if A is a constrictor for the semigroup $(T^n)_{n=0}^\infty$, and we write $A \in \operatorname{Constr}_{\|\cdot\|}(T)$ instead of $A \in \operatorname{Constr}_{\|\cdot\|}\Big((T^n)_{n=0}^\infty\Big)$.

Lemma 1.3.2. *If $\mathcal{T}$ possesses a (weakly) compact constrictor, then $\mathcal{T}$ is (weakly) almost periodic.*

Proof. Let $A \in \operatorname{Constr}(\mathcal{T})$ be (weakly) compact. Given an element $x \in B_X$ and a sequence $(t_n)_{n=0}^\infty$ in J that converges to ∞, take for any t_n an $a_{t_n} \in A$ such that $\|a_{t_n} - T_{t_n}x\| \to 0$. The sequence $(a_{t_n})_{n=0}^\infty$ has a (weakly) convergent subsequence $(a_{s_n})_{n=0}^\infty$, since A is (weakly) compact. Then the subsequence $(T_{s_n}x)_{n=0}^\infty$ of $(T_{t_n}x)_{n=0}^\infty$ is also (weakly) convergent to the same limit. □

1.3.2 The following theorem is due to Lasota, Li, and Yorke [70] for Markov operators in L^1-spaces, and to Phòng [94] (cf. also [97]) and Sine [120] in the general case.

Theorem 1.3.3 (Lasota–Li–Yorke–Phòng–Sine). *Given a one-parameter bounded semigroup $\mathcal{T}$ in a Banach space X, the following assertions are equivalent:*

(i) *there exists a compact $A \in \operatorname{Constr}_{\|\cdot\|}(\mathcal{T})$;*

(ii) *there exists a $\mathcal{T}$-reducing decomposition $X = X_0(\mathcal{T}) \oplus X_r(\mathcal{T})$ with*

$$X_0(\mathcal{T}) = \{x \in X : \lim_{t\to\infty} \|T_t x\| = 0\} \quad \textit{and} \quad \dim(X_r(\mathcal{T})) < \infty\,.$$

Proof. (i) $\Rightarrow$ (ii): The semigroup $\mathcal{T}$ is almost periodic by Lemma 1.3.2. Applying Theorem 1.1.4, obtain the decomposition $X := X_0(\mathcal{T}) \oplus X_r(\mathcal{T})$ onto $\mathcal{T}$-invariant subspaces $X_0(\mathcal{T})$ and $X_r(\mathcal{T})$ such that any $x \in X_r(T)$ is a norm-cluster point of the orbit $\{T_t x\}_{t\in J}$. Thus $B_{X_r(\mathcal{T})} \subseteq A$. Henceforth $B_{X_r(\mathcal{T})}$ is norm compact and then $\dim(X_1) < \infty$.

(ii) $\Rightarrow$ (i): Assume that $X = X_0(\mathcal{T}) \oplus X_r(\mathcal{T})$ is a $\mathcal{T}$-reducing decomposition with $X_0(\mathcal{T}) = \{x \in X : \lim_{t\to\infty} \|T_t x\| = 0\}$ and $X_r(\mathcal{T})$, $\dim(X_r(\mathcal{T})) < \infty$. Then the set $M_{\mathcal{T}}\|P\|B_{X_r(\mathcal{T})}$ is a compact constrictor for $\mathcal{T}$, where $M_{\mathcal{T}} = \sup_{t\in J} \|T_t\|$, and $P \in \mathcal{L}(X)$ is a projection that satisfies $P(X) = X_r(\mathcal{T})$ and $\ker P = X_0(\mathcal{T})$. □

Definition 1.3.4. A one-parameter bounded semigroup $\mathcal{T}$ is called *constrictive* if $\mathcal{T}$ satisfies the conditions of Theorem 1.3.3. An operator $T \in \mathcal{L}(X)$ is called *constrictive* whenever the semigroup $(T^n)_{n=1}^{\infty}$ satisfies the same property.

Theorem 1.3.3 shows that constrictive semigroups can be asymptotically investigated with methods of linear algebra. This remark asserts the importance of constrictive semigroups. Unfortunately, with exception of several rather special cases, it is difficult to check whenever a semigroup $\mathcal{T}$ is constrictive. Therefore, the problem arises:

to relax conditions on a semigroup $\mathcal{T}$ to be constrictive.

This problem will be central in this and several other sections of our book.

Due to the uniform boundedness principle, any constrictive semigroup is automatically bounded. For this reason, we mainly consider bounded semigroups.

1.3.3 Let X be a Banach space and let $\mathcal{T} = (T_t)_{t\in J}$ be a one-parameter bounded semigroup in X, then

$$X_0(\mathcal{T}) = \{x \in X : \lim_{t\to\infty} \|T_t x\| = 0\}$$

is, obviously, a closed *$\mathcal{T}$-invariant* subspace in X. Whenever $X_0(\mathcal{T})$ possesses a $\mathcal{T}$-invariant supplement, the semigroup $\mathcal{T}$ satisfies the conditions of Theorem 1.3.3 and, henceforth, $\mathcal{T}$ is constrictive. This remark is a key to the following definition.

Definition 1.3.5. We call a one-parameter bounded semigroup $\mathcal{T}$ *quasi-constrictive* if

$$\mathrm{codim} X_0(\mathcal{T}) < \infty.$$

An operator $T \in \mathcal{L}(X)$ is called *quasi-constrictive* whenever the semigroup $(T^n)_{n=1}^{\infty}$ satisfies the same property.

The following result is obvious.

Proposition 1.3.6. *Given a one-parameter bounded semigroup $\mathcal{T} = (T_t)_{t \in J}$. Then $\mathcal{T}$ is quasi-constrictive if and only if the operator T_τ is quasi-constrictive for some $\tau \in J$.* □

This proposition allows us to consider in the investigation of one-parameter quasi-constrictive semigroups the discrete case only.

1.3.4 Before studying quasi-constrictive operators in more detail, let us give several examples.

Example 1.3.7. Let $X := C[0,1]$. Define $T : X \to X$ by $Tf(t) := tf(t)$. Then

$$X_0(T) = \{f \in C[0,1] : \lim_{n\to\infty} \|T^n f\| = 0\} = \{f \in C[0,1] : f(1) = 0\}$$

is closed and has co-dimension 1. So T is quasi-constrictive, but it is not constrictive, since it has no non-trivial eigenvector.

Example 1.3.8. Let $X = c_0$ with the sup-norm $\|\cdot\|$. Denote by e_k the element of X, the k-th coordinate of which is equal to 1 and all other coordinates are zero. Fix $\alpha \in \mathbb{C}$ and define the operator $S_\alpha : X \to X$,

$$S_\alpha(e_k) = \begin{cases} e_1 + \alpha\, e_2 & k = 1 \\ e_{k+1} & \text{else} \end{cases}.$$

Set $T_\alpha := (I + S_\alpha)/2$. Obviously, T_α is power bounded (moreover, it is contractive if $|\alpha| \le 1$). For $k \ge 2$, we have

$$T_\alpha^n(e_k) = 2^{-n} \sum_{l=0}^{n} \binom{n}{l} e_{k+l}.$$

So $\|T_\alpha^n(e_k)\| = 2^{-n}\binom{n}{[n/2]}$, where $[q]$ is the integer part of q. But

$$2^{-n}\binom{n}{[n/2]} \sim 1/\sqrt{\pi[n/2]},$$

as $n \to \infty$. So $T_\alpha^n(e_k)$ converges to 0 for all $k \ge 2$. On the other hand, $\|T_\alpha^n(e_1)\| \ge 1$ for all n. Hence

$$X_0(T_\alpha) := \{x \in c_0 : x_0 = 0\},$$

and T_α is quasi-constrictive. Obviously, T_α is constrictive if and only if $\alpha = 0$.

Example 1.3.9. Consider a generalization of Example 1.3.8. Let $0 \ne S \in \mathcal{L}(X)$ be a power bounded operator such that $\operatorname{codim}(\overline{(I - S)X}) < \infty$. Without any restriction, we may assume $\|S\| = 1$. Take some real $\alpha \in (0,1)$ and consider the operator $T_\alpha := \alpha \cdot I + (1-\alpha)\cdot S$. Theorem 1.1.22 (or the result of Foguel and Weiss [44, *Lm.*2.1]) implies

$$\|T_\alpha^{n+1} - T_\alpha^n\| = \|T_\alpha^n \circ (T_\alpha - I)\| = \|(1-\alpha)\cdot T_\alpha^n \circ (I - S)\| \to 0.$$

Then $\overline{(I-S)X} \subseteq X_0(T) = \{x \in X : \lim_{n\to\infty} \|T_\alpha^n x\| = 0\}$, and

$$\operatorname{codim}(X_0(T)) \leq \operatorname{codim}\overline{((I-S)X)} < \infty.$$

Consequently, T_α is quasi-constrictive.

Example 1.3.10. A Markov operator T in $L^1(\Omega, \Sigma, \mu)$ is called *completely mixing* (see, for example, [67, 8.1]), whenever $\lim_{n\to\infty} \|T^n f\| = 0$ for all $f \in L^1(\Omega, \Sigma, \mu)$ such that $\|f_+\| = \|f_-\|$, where $f_+(x) = \max(f(x), 0)$ a.e. and $f_- = (-f)_+$. Obviously, every completely mixing Markov operator is quasi-constrictive, and it is constrictive if and only if it possesses a non-trivial fixed vector.

Example 1.3.11. Let T be a quasi-constrictive operator. Let $X = X_0 \oplus Y$, where Y is an arbitrary finite-dimensional complement of X_0. Let P be a projection from X onto Y with kernel X_0, and define

$$Q = I - P, \quad U = Q \circ T \circ Q, \quad V = Q \circ T \circ P, \quad W = P \circ T \circ P.$$

Since X_0 is T-invariant, $P \circ T \circ Q = 0$, hence $T = U + V + W$ or, in the matrix form,

$$T = \begin{bmatrix} U & V \\ 0 & W \end{bmatrix}.$$

Then T^n is represented by

$$T^n = \begin{bmatrix} U^n & \sum_{k=0}^{n-1} U^k \circ V \circ W^{n-1-k} \\ 0 & W^n \end{bmatrix}.$$

Let $\rho(S)$ denote the resolvent set $\mathbb{C} \setminus \sigma(S)$ of the operator S. Then an easy calculation shows that, for $\lambda \in \rho(U) \cap \rho(W)$,

$$(\lambda - T)^{-1} = \begin{bmatrix} (\lambda - U)^{-1} & (\lambda - U)^{-1} \circ V \circ (\lambda - W)^{-1} \\ 0 & (\lambda - W)^{-1} \end{bmatrix}$$

holds. In particular, $\sigma(T) \subseteq \sigma(U) \cup \sigma(W)$. Obviously, T is constrictive if $V = 0$.

1.3.5 Two problems arise in connection with Theorem 1.3.3. One of them was mentioned above, and the other one is the following:

Find an appropriate analogue of Theorem 1.3.3 *for quasi-constrictive semigroups.*

We study these questions in this section. First of all we need a special notion.

Definition 1.3.12. Let $A \subseteq X$, then the *Hausdorff measure of non-compactness* $\chi_{\|\cdot\|}(A)$ of A is defined as:

$$\chi_{\|\cdot\|}(A) := \inf\{\alpha \geq 0 : A \subseteq \bigcup_{i=1}^{p} B(x_i, \alpha); p \in \mathbb{N}, \overline{x_1, x_p} \in X\},$$

where $B(x, \alpha)$ denotes the closed ball centered at x with radius α.

Of course, in general, $\chi_{\|\cdot\|}(A)$ depends on the choice of the norm $\|\cdot\|$, but it is clear that a set A satisfies $\chi_{\|\cdot\|}(A)=0$ for 1, and then for any other equivalent norm on X, if and only if A is conditionally compact. The following example explains the situation with measure of non-compactness of constrictors.

Example 1.3.13. Let $\alpha \geq 0$ and let T_α be defined on c_0 as in Example 1.3.8. Then the operator T_α has a constrictor

$$A_\alpha := [-e_1, e_1] + \alpha B_X$$

such that $\chi_{\|\cdot\|}(A_\alpha) = \alpha$; and, for every $A \in \mathrm{Constr}_{\|\cdot\|}(T_\alpha)$, we have $\chi_{\|\cdot\|}(A) \geq \alpha$, since the sequence $T_\alpha^n(e_1)$ is increasing and its supremum in ℓ^∞ is easily determined as $(1, \alpha, \alpha, \ldots \alpha, \ldots)$. This implies also that when $\alpha > 0$ and $\lambda \in \mathbb{C}$, $|\lambda| = 1$, the operator $\lambda T_{_\alpha}$ is mean ergodic if and only if $\lambda \notin \sigma_\pi(T_\alpha) = \{1\}$.

Take some real $\beta > 0$ and consider the equivalent norm $\|\cdot\|_\beta$ on c_0 defined as

$$\|x\|_\beta := \sup\Big\{|x_1|, \beta\|x - x_1 e_1\|\Big\}.$$

It is easy to see that $\chi_{\|\cdot\|_\beta}(A) \geq \alpha\cdot\beta$ holds for every $A \in \mathrm{Constr}_{\|\cdot\|_\beta}(T_\alpha)$. In particular, if $\beta = 1/\alpha$, then the operator T_α has no constrictor A satisfying $\chi_{\|\cdot\|_\beta}(A) < 1$. It should be noted that the operator T_α is a contraction with respect to the norm $\|\cdot\|_\beta$ whenever $\beta \leq 1/\alpha$.

1.3.6 Now we state the central theorem about quasi-constrictive operators, which plays the analogous role in the quasi-constrictive case as the Lasota–Li–Yorke–Phòng–Sine theorem in the constrictive case.

Theorem 1.3.14 (Emel'yanov–Wolff). *Let T be a power bounded operator in a Banach space X. Then the following conditions are equivalent*:

(i) *T is quasi-constrictive*;

(ii) *for every $\varepsilon > 0$, there exist an equivalent norm $\|\cdot\|_\varepsilon$ on X and $A_\varepsilon \in \mathrm{Constr}_{\|\cdot\|_\varepsilon}(T)$ such that $\chi_{\|\cdot\|_\varepsilon}(A_\varepsilon) \leq \varepsilon$*;

(iii) *there exist an equivalent norm $\|\cdot\|_1$ on X and $A \in \mathrm{Constr}_{\|\cdot\|_1}(T)$ such that $\chi_{\|\cdot\|_1}(A) < 1$.*

For proving it, we need the following result of Krasnoselskii, Krein, and Milman [66], which can also be found in Kato's book [60, Ch. IV, 2.2], in Day [21], or in Lindenstrauss and Tzafriri [78, Lemma 2.c.8].

Lemma 1.3.15 (Krasnoselskii–Krein–Milman). *Let Y and Z be subspaces of a Banach space X with $\dim Y < \infty$ and $\dim Z > \dim Y$, then there exists $0 \neq z \in Z$ such that $\|z\| = \mathrm{dist}(z, Y)$.* □

Proof of Theorem 1.3.14. (i) ⇒ (ii): Take some $\varepsilon > 0$ and let P_0 be a projection of X onto X_0. Let $P := I - P_0$ and let $C := \sup\{\|T^n x\| : x \in B_{P(X)}, n \geq 0\}$. Then

$$\begin{aligned}\limsup_{n\to\infty} \|T^{n+m}y\| &\leq \limsup_{n\to\infty} \|T^n \circ P \circ T^m y\| \\ &\quad + \limsup_{n\to\infty} \|T^n \circ (P - I) \circ T^m y\| \\ &\leq C\|P \circ T^m y\|\end{aligned}$$

for all $m \in \mathbb{N}$, $y \in X$. Consider the equivalent norm $\|\cdot\|_\varepsilon$ on X:

$$\|y\|_\varepsilon := \|Py\| + \varepsilon C^{-1}\|P_0\|^{-1}\|P_0 y\| \qquad (y \in X).$$

Take $y \in X$, $\|y\|_\varepsilon \leq 1$. It is easy to see that

$$\limsup_{n\to\infty} \operatorname{dist}(P \circ T^n \circ P\, y, CP(B_X)) = 0$$

in each of the equivalent norms on X. Consequently

$$\begin{aligned}\limsup_{n\to\infty} \operatorname{dist}_{\|\cdot\|_\varepsilon}(T^n y, CP(B_X)) &\leq \limsup_{n\to\infty} \|T^n y - P \circ T^n \circ P\, y\|_\varepsilon \\ &\leq \limsup_{n\to\infty} \|T^n \circ P_0\, y\|_\varepsilon \\ &\quad + \limsup_{n\to\infty} \|T^n \circ P\, y - P \circ T^n \circ P\, y\|_\varepsilon \\ &= \limsup_{n\to\infty} \|P_0 \circ T^n \circ P\, y\|_\varepsilon \\ &= \varepsilon C^{-1}\|P_0\|^{-1} \limsup_{n\to\infty} \|P_0 \circ T^n \circ P\, y\| \\ &\leq \varepsilon\|Py\| \leq \varepsilon\|y\|_\varepsilon \leq \varepsilon.\end{aligned}$$

The set $CP(B_X)$ is compact since $\dim P(X) < \infty$. So, we obtain that

$$A_\varepsilon := CP(B_X) + \{y \in X : \|y\|_\varepsilon \leq \varepsilon\}$$

is a constrictor for the operator T such that $\chi_{\|\cdot\|_\varepsilon}(A_\varepsilon) \leq \varepsilon$.

(ii) ⇒ (iii): It is obvious.

(iii) ⇒ (i): Without any restriction, we may assume that $\|\cdot\|_1$ is the initial norm $\|\cdot\|$ on X. Take a free ultra-filter $\mathcal{U}$ on $\mathbb{N}$ and consider the bounded ultra-power $X_{\mathcal{U}} := \ell^\infty_{\mathcal{U}}(X)/c_{\mathcal{U}}(X)$ according to **1.3.16**. Then $X_{\mathcal{U}}$ is a Banach space with respect to the norm

$$\|\widehat{(x_n)}\| := \lim_{\mathcal{U}} \|x_n\|.$$

We identify X with the subspace in $X_{\mathcal{U}}$ of all equivalence classes of constant sequences. Define the linear operator $\mathbf{T} : X \to X_{\mathcal{U}}$ by

$$\mathbf{T}x := \widehat{(T^n x)} \quad (\forall x \in X).$$

Operator $\mathbf{T}$ is easily seen to be bounded.

Using the fact that $\chi(A) < 1$, take a real $\delta > 0$ such that $(1+\delta)\alpha < 1$, where a real α satisfies $\chi(A) < \alpha < 1$, and take a finite set $\{a_i\}_{i=1}^p \subseteq X$ which satisfies $A \subseteq \bigcup_{i=1}^p B(a_i, \alpha)$.

Let $x \in B_{\mathbf{T}(X)}$ be arbitrary. Then $x = \mathbf{T}y$ for some $y \in X$ such that

$$\{n : \|T^n y\| < 1 + \frac{\delta}{2}\} \in \mathcal{U}.$$

Consequently, there exists $m \in \mathbb{N}$ satisfying $\|T^m y\| < 1 + \frac{\delta}{2}$. Since

$$\lim_{n\to\infty} \operatorname{dist}\Big(T^n \circ T^m\, y, \bigcup_{i=1}^p B(b_i, (1+\frac{\delta}{2})\alpha)\Big) = 0\,,$$

where $b_i := (1+\frac{\delta}{2})a_i$, there exists $i_x \in \overline{1,p}$ such that

$$\{k \in \mathbb{N} : \|T^k y - b_{i_x}\| < (1+\delta)\alpha\} \in \mathcal{U}.$$

So, we have that $\|x - b_{i_x}\| \leq (1+\delta)\alpha$ holds in $X_{\mathcal{U}}$. Consequently,

$$B_{\mathbf{T}(X)} \subseteq \bigcup_{i=1}^p B(b_i, (1+\delta)\alpha)),$$

and Lemma 1.3.15 implies $\dim \mathbf{T}(X) \leq p < \infty$ due to $(1+\delta)\alpha < 1$.

Assume $\lim_{\mathcal{U}} \|T^n x\| = 0$. Since the operator T is power bounded, it follows that $\lim\limits_{n\to\infty} \|T^n x\| = 0$. This proves that

$$\ker \mathbf{T} \subseteq X_0(T) = \{x \in X : \lim_{n\to\infty} \|T^n x\| = 0\}.$$

The inverse inclusion is trivial. The equality $X_0(T) = \ker \mathbf{T}$ implies

$$\operatorname{codim} X_0(T) = \dim \mathbf{T}(X) < \infty.$$

Consequently, T is quasi-constrictive. $\square$

In the strongly continuous case, Theorem 1.3.14 implies easily the following result.

Corollary 1.3.16. *Let $\mathcal{T} = (T_t)_{t\geq 0}$ be a bounded C_0-semigroup in X. Then the following conditions are equivalent:*

(i) *$\mathcal{T}$ is quasi-constrictive;*

(ii) *for every $\varepsilon > 0$, there exist an equivalent norm $\|\cdot\|_\varepsilon$ on X and $A_\varepsilon \in \mathrm{Constr}_{\|\cdot\|_\varepsilon}(\mathcal{T})$ such that $\chi_{\|\cdot\|_\varepsilon}(A_\varepsilon) \leq \varepsilon$;*

(iii) *there exist an equivalent norm $\|\cdot\|_1$ on X and $A \in \mathrm{Constr}_{\|\cdot\|_1}(\mathcal{T})$ such that $\chi_{\|\cdot\|_1}(A) < 1$.* $\square$

1.3.7 In the discrete case, we can say even more about conditions under which a semigroup is quasi-constrictive.

Definition 1.3.17. Let $T \in \mathcal{L}(X)$. A subspace Y of X is called *ε-T-invariant* whenever

$$\operatorname{dist}(Ty, Y) \leq \varepsilon \|Ty\|$$

for all $y \in Y$.

The following result is an analogue of Theorem 1.3.3 in some reasonable sense.

Theorem 1.3.18 (Emel'yanov–Wolff). *Let T be a power bounded operator in a Banach space X. Then the following conditions are equivalent*:

(i) *T is quasi-constrictive*;

(ii) *for every $\varepsilon > 0$, there exists a finite-dimensional ε-T-invariant subspace Y with $X = X_0 \oplus Y$*;

(iii) *for every finite-dimensional subspace Y with $X = X_0 \oplus Y$ and for every $\varepsilon > 0$, there exists $n \in \mathbb{N}$ such that $T^n(Y)$ is ε-T-invariant and $X = X_0 \oplus T^n(Y)$.*

Proof. The implications (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) are obvious.

(i) $\Rightarrow$ (iii): Let $X_{\mathcal{U}}$ and $\mathbf{T} : X \to X_{\mathcal{U}}$ be defined as in the proof of Theorem 1.3.14. Let Y be an arbitrary algebraic complement of $X_0(T)$ in X. Obviously,

$$T^n(Y) \oplus X_0(T) = X$$

holds for all $n \in \mathbb{N}$.

Now, let $\varepsilon > 0$ be given. If the assertion does not hold, then for every $n \in \mathbb{N}$ there exists a normalized $y_n \in Y$ such that

$$\operatorname{dist}(T(T^n y_n), T^n(Y)) \geq \varepsilon \|T^{n+1} y_n\|.$$

The unit sphere of Y is compact, so $\lim_{\mathcal{U}} y_n = y$ exists. But then

$$\begin{aligned} \lim_{\mathcal{U}} \operatorname{dist}(T^{n+1}y, T^n(Y)) &= \lim_{\mathcal{U}} \operatorname{dist}(T^{n+1}y_n, T^n(Y)) \\ &\geq \varepsilon \cdot \lim_{\mathcal{U}} \|T^{n+1}y_n\| \\ &= \varepsilon \cdot \lim_{\mathcal{U}} \|T^{n+1}y\| > 0 \end{aligned}$$

holds, since T is power bounded and $y \notin X_0 = \ker \mathbf{T}$. We define $\hat{T}$ in $X_{\mathcal{U}}$ by $\hat{T}\widehat{(x_n)} = \widehat{(Tx_n)}$ and obtain

$$\operatorname{dist}(\hat{T} \circ \mathbf{T}(y), \mathbf{T}(Y)) \geq \lim_{\mathcal{U}} \operatorname{dist}(T^{n+1}y, T^n(Y)) > 0.$$

But

$$\hat{T} \circ \mathbf{T}(y) = \widehat{(T^{n+1}y)} = \mathbf{T} \circ T(y) \in \mathbf{T}(X) = \mathbf{T}(Y),$$

a contradiction. □

1.3.8 Now we present some conditions on a quasi-constrictive semigroup to be constrictive. We begin with a discrete semigroup.

Theorem 1.3.19 (Emel'yanov–Wolff). *Let X be a Banach space and $T \in \mathcal{L}(X)$. Then the following assertions are equivalent*:

(i) *T is weakly almost periodic and quasi-constrictive*;

(ii) *T is constrictive.*

Proof. (ii) $\Rightarrow$ (i): It follows from Theorem 1.3.3 and from Lemma 1.3.2.

(i) $\Rightarrow$ (ii): Let $P \in L(X)$ be a finite rank projection such that

$$\ker P = X_0(T) = \{x \in X : \lim_{n\to\infty} \|T^n x\| = 0\}.$$

Take some free ultra-filter $\mathcal{U}$ on $\mathbb{N}$. It is easy to see that, by

$$Wx := w - \lim_{\mathcal{U}} T^n x \quad (x \in X),$$

there is defined a linear operator W in X. Here we use the fact that the weak limit $w\text{-}\lim_{\mathcal{U}} T^n x$ along $\mathcal{U}$ exists (see **1.3.16**) since T is weakly almost periodic. Since T is power bounded, the operator W is bounded. Obviously, $X_0(T) \subseteq \ker(W)$, so $W(X) = W \circ P(X)$.

First of all we show that $W(X) \cap X_0(T) = \{0\}$. Let $Wu \in X_0(T)$ be arbitrary. Since P has finite rank,

$$\begin{aligned} \|\cdot\|\text{-}\lim_{\mathcal{U}} P \circ T^n u &= w\text{-}\lim_{\mathcal{U}} P \circ T^n u \\ &= P(w\text{-}\lim_{\mathcal{U}} T^n u) \\ &= P \circ W u \\ &= 0 \end{aligned}$$

holds. Thus to every $\varepsilon > 0$, there exists $U_\varepsilon \in \mathcal{U}$ such that $\|P \circ T^m u\| \le \varepsilon M_T^{-1}$ for all $m \in U_\varepsilon$, where $M_T := \sup_{n\ge 0} \|T^n\|$. Remark that

$$\begin{aligned} \limsup_{n\to\infty} \|T^{n+m} y\| &\le \limsup_{n\to\infty} \|T^n \circ P \circ T^m y\| + \limsup_{n\to\infty} \|T^n \circ (P - I) \circ T^m y\| \\ &\le M_T \|P \circ T^m y\| \end{aligned}$$

for all $m \in \mathbb{N}$, $y \in X$. So, if we take $m \in U_\varepsilon$ we obtain

$$\begin{aligned} \limsup_{n\to\infty} \|T^n u\| &= \limsup_{n\to\infty} \|T^{m+n} u\| \\ &\le M_T \|P \circ T^m u\| \\ &\le \varepsilon\,. \end{aligned}$$

Consequently, $Wu = 0$ and the assertion $W(X) \cap X_0(T) = \{0\}$ is proved.

By Theorem 1.3.14, we may assume (by passing, if it is necessary, to an equivalent norm) that there is a constrictor $A \subseteq X$ of T such that $\chi(A) < 1/3$. Let $X_{\mathcal{U}}$ and $\mathbf{T} : X \to X_{\mathcal{U}}$ be defined as in the proof of Theorem 1.3.14. Assume $\dim W(X) < \operatorname{codim} X_0(T)$, then $\ker \mathbf{T} = X_0(T)$ is a proper subspace of $\ker W$ and, therefore, there exists $y = \mathbf{T}x \in \mathbf{T}(X)$ such that $\|y\| = 1$ and $Wx = 0$. By the definition of $\mathbf{T}$, $\limsup\limits_{n\to\infty} \operatorname{dist}(T^n x, A) < 1/3$. Hence there exists $z \in A$ for which $\|z - y\| < 1/3$ holds in $X_{\mathcal{U}}$. Then

$$\{n : \|z - T^n x\| < 1/3\} \in \mathcal{U}.$$

Consequently,

$$\|z - (w - \lim_{\mathcal{U}} T^n x)\| \leq 1/3$$

and

$$\|z\| \geq \|y\| - \|z - y\| > 2/3$$

since $\|y\| = 1$. This implies $Wx = w - \lim_{\mathcal{U}} T^n x \neq 0$. The contradiction shows that

$$\dim W(X) = \operatorname{codim} X_0(T).$$

Together with $W(X) \cap X_0(T) = \{0\}$, this implies $X = X_0(T) \oplus W(X)$.

Finally, let us show that $W(X)$ is T-invariant. Let $Wx \in W(X)$ be arbitrary, then

$$\begin{aligned} T(Wx) = T(w\text{-}\lim_{\mathcal{U}} T^n x) &= w\text{-}\lim_{\mathcal{U}} T(T^n x) \\ &= w\text{-}\lim_{\mathcal{U}} T^n (Tx) \\ &= W(Tx) \end{aligned}$$

by the weak continuity of T. This proves the T-invariance of $W(X)$, and so T is constrictive. $\square$

There is a variant of Theorem 1.3.19 for a C_0-semigroup which is an easy corollary of Theorem 1.3.19. We leave its proof to the reader.

Corollary 1.3.20. *Let $\mathcal{T} = (T_t)_{t\geq 0}$ be a bounded C_0-semigroup in a Banach space X. Then the following conditions are equivalent*:

(i) *$\mathcal{T}$ is weakly almost periodic and quasi-constrictive*;

(ii) *$\mathcal{T}$ is constrictive.* $\square$

The following two corollaries of Theorem 1.3.19 are obvious.

Corollary 1.3.21. *Let $\mathcal{T} = (T_t)_{t\in J}$ be a one-parameter weakly almost periodic semigroup. Then $\mathcal{T}$ is constrictive if and only if, for some equivalent norm $\|\cdot\|_1$ the semigroup $\mathcal{T}$ possesses a constrictor A which satisfies $\chi_{\|\cdot\|_1}(A) < 1$.* $\square$

Corollary 1.3.22. *Let $\mathcal{T} = (T_t)_{t\in J}$ be a one-parameter bounded semigroup in a reflexive Banach space X. Then $\mathcal{T}$ is constrictive if and only if for some equivalent norm $\|\cdot\|_1$ the semigroup $\mathcal{T}$ possesses a constrictor A which satisfies $\chi_{\|\cdot\|_1}(A) < 1$.* $\square$

Related Results and Notes

1.3.9 All results of this section are taken from papers [31], [35], [38], [94], and [120]. In some papers, the word *attractor* is also used instead of *constrictor*. Lasota, Li, and Yorke have shown in [70] Theorem 1.3.3 for the special but important case of Markov operators in $L_1(\Omega, \Sigma, \mu)$. Then Phòng [94] and independently Sine [120] have showed that the generalization of the Lasota–Li–Yorke result, given by Theorem 1.3.3, is a simple corollary of the Jacobs–Deleeuw–Glicksberg theorem. In Phòng's paper [94], Theorem 1.3.3 is given in a slightly more general form, for a *representation* of an abelian semigroup $\mathbb{P}$ in $\mathcal{L}(X)$, which is a homomorphism from $\mathbb{P}$ to $\mathcal{L}(X)$ equipped with the operation of the composition of operators. A representation $(T_t)_{t\in\mathbb{P}}$ is called *bounded* if $\sup_{t\in\mathbb{P}} \|T_t\| < \infty$. Every abelian semigroup $\mathbb{P}$ is ordered in the following way. A natural pre-order $\succeq$ on $\mathbb{P}$ (see, for example, [67, p. 75], [81]) is given by

$$t \succeq s \quad \Leftrightarrow \quad (t = s \text{ or } \exists u \in \mathbb{P} : t = s + u).$$

Set $\mathbb{P}_t = \{t + u : u \in \mathbb{P}\}$. Since $\mathbb{P}_s \cap \mathbb{P}_t \supseteq \mathbb{P}_{s+t}$, the sections $\mathbb{P}_t$ generate a filter $\mathcal{F}$. We shall write $\lim\limits_{t\to\infty}$ instead of $\lim_{\mathcal{F}}$. Similarly, $\overline{\lim\limits_{t\to\infty}} \|T_t x\| = \alpha$ means that to every $t \in \mathbb{P}$ and $\varepsilon > 0$ there exists $t(\varepsilon) \succeq t$ such that

$$\alpha - \varepsilon < \sup\{\|T_s x\| : s \in \mathbb{P}_{t(\varepsilon)}\} < \alpha + \varepsilon .$$

Definition 1.3.23. Let $\mathbb{P}$ be an abelian semigroup, let X be a Banach space with the norm $\|\cdot\|$, and let $\mathcal{T} = (T_t)_{t\in\mathbb{P}}$ be a representation of an abelian semigroup $\mathbb{P}$ in $\mathcal{L}(X)$. A subset $A \subseteq X$ is called a *constrictor* for $\mathcal{T}$ if

$$\lim_{t\to\infty} \operatorname{dist}_{\|\cdot\|}(T_t x, A) = 0 \qquad (\forall x \in B_X) .$$

Definition 1.3.24. A representation $\mathcal{T} = (T_t)_{t\in\mathbb{P}}$ of an abelian semigroup $\mathbb{P}$ in $\mathcal{L}(X)$ is called *constrictive* if $\mathcal{T}$ has a compact constrictor.

The proof of the following theorem is similar to the proof of Theorem 1.3.3.

Theorem 1.3.25 (Phòng). *Given a bounded representation $\mathcal{T} = (T_t)_{t\in\mathbb{P}}$ of an abelian semigroup $\mathbb{P}$ in $\mathcal{L}(X)$, the following assertions are equivalent*:

(i) *there exists a compact constrictor for $\mathcal{T}$*;

(ii) *there exists a $\mathcal{T}$-reducing decomposition $X = X_0(\mathcal{T}) \oplus X_r(\mathcal{T})$ with*

$$X_0(\mathcal{T}) = \{x \in X : \lim_{t\to\infty} \|T_t x\| = 0\} \quad \textit{and} \quad \dim(X_r) < \infty. \qquad \square$$

Exercise 1.3.26. Show that Theorem 1.3.3 can be proved as a corollary of Theorems 1.3.14 and 1.3.19 without using the Jacobs–Deleeuw–Glicksberg theorem.

It is interesting that we did not use the Jacobs–Deleeuw–Glicksberg theorem in the proofs of Theorems 1.3.14 and 1.3.19, however other deep techniques, like ultra-powers and the Krasnoselskii–Krein–Milman lemma, have been used there.

1.3.10 Examples 1.3.7, 1.3.8, 1.3.9, 1.3.10, 1.3.11, and 1.3.13 are taken from [34], [35], and [50]. The strongly continuous version of Example 1.3.11 follows.

Let $(U_t)_{t\geq 0}$ be a C_0-semigroup in the Banach space Z which converges strongly to 0 as $t \to \infty$. Let Y be a finite-dimensional normed space. Let B be a linear operator in Y and set $W_t = \exp(tB)$. Finally, let $V \in \mathcal{L}(Y, Z)$. Consider the semigroup $\mathcal{T} = (T_t)_{t\geq 0}$ in $X = Z \oplus Y$ given by

$$T_t = \begin{bmatrix} U_t & \int\limits_0^t U_{t-s} \circ V \circ W_s \, ds \\ 0 & W_t \end{bmatrix}.$$

It can be shown that this semigroup is quasi-constricted and $X_0(\mathcal{T})$ contains the closed space Z which has finite co-dimension in X_0.

The dilation schema presented in **1.1.13** preserves the property of a semigroup to possess a constrictor of the special type.

Exercise 1.3.27. Let X and Y be Banach spaces, $\dim Y < \infty$, and let $R \in \mathcal{L}(X, Y)$. Show that an operator semigroup $\mathcal{T}$ in X possesses a (weakly) compact constrictor if and only if the semigroup $\pi_R(\mathcal{T})$ in $X \times Y$ has the same property.

Exercise 1.3.28. Let X and Y be Banach spaces, $\dim Y < \infty$, and let $R \in \mathcal{L}(X, Y)$. Show that an operator semigroup $\mathcal{T}$ in X is quasi-constrictive if and only if the semigroup $\pi_R(\mathcal{T})$ in $X \times Y$ is quasi-constrictive.

Power bounded quasi-compact operators is another important class of constrictive operators. More constrictive operators will be studied in Sections 2.1, 2.2, and 3.3. We show there that, in many cases, for obtaining a decomposition $X = X_0(\mathcal{T}) \oplus X_r$ as in Theorem 1.3.3, it is enough to find in some sense small (but not necessarily of Hausdorff measure of non-compactness less then 1) constrictor. The idea to use Lemma 1.3.2 in the proof of Theorem 1.3.14 was suggested by Troitsky.

1.3.11 The following important extension of Theorem 1.3.3 for a one-parameter operator semigroup was obtained recently by Storozhuk [124]. This result seems to be true also for any bounded operator representation of any abelian semigroup. It was proved in [124] directly (without using the Lasota–Li–Yorke–Phòng–Sine theorem and without any kind of argument exploiting the Jacobs–Deleeuw–Glicksberg theorem) that under the condition (1.46) there exists a T-reducing decomposition $X = X_0(T) \oplus X_r(T)$ as in Theorem 1.3.3(ii).

Theorem 1.3.29 (Storozhuk). *Let $K \subseteq X$ be a compact subset of a real or complex Banach space X, and let $T \in \mathcal{L}(X)$ be a power bounded operator such that*

$$\lim\inf_{n\to\infty} \operatorname{dist}(T^n x, K) = 0 \quad (\forall x \in B_X), \tag{1.46}$$

then T is constrictive.

Before we prove this theorem, we need the following lemma.

Lemma 1.3.30. *Let $K \subseteq X$ be a compact subset of a real or complex Banach space X, and let $T \in \mathcal{L}(X)$ be a surjective isometry such that*

$$\lim_{n\to\infty}\inf \operatorname{dist}(T^n x, K) = 0 \quad (\forall x \in B_X)\,. \tag{1.47}$$

Then $\dim(X) < \infty$.

Proof. Assume $\dim(X) = \infty$.
I. The complex case: Since T is a surjective isometry, $\sigma(T)$ is a subset of the unit circle Γ of $\mathbb{C}$. It follows easily from (1.47), that every normalized eigenvector x belongs to the compact set

$$K^\circ := \{\xi a : \xi \in \mathbb{C}\,, |\xi| = 1\,, a \in K\}\,.$$

Therefore $\ker(\lambda I - T) \cap B_X \subseteq K^\circ$ and $\dim\ker(\lambda I - T) < \infty$ for any $\lambda \in \sigma(T)$. Moreover, there are only finitely many mutually distinct eigenvalues in $\sigma(T)$. Otherwise, by Lemma 1.3.15, there exists a sequence $(x_n)_{n=1}^\infty$ of normalized eigenvectors correspondent to mutually distinct eigenvalues, which satisfies $\|x_n - x_m\| \geq 1$ for all $n \neq m$. This contradicts the compactness of K°, since $x_n \in K^\circ$ for every n. Consequently, if $\sigma(T)$ consists of eigenvalues only, then X is the direct sum of finite number of finite-dimensional eigenspaces correspondent to distinct eigenvalues, which contradicts our assumption that $\dim(X) = \infty$.

So, there exists $\lambda \in \sigma(T)$ which is not an eigenvalue of T. It belongs to the approximative spectrum of T (see, for example, [2, Thm.6.18]). Therefore, there exists a normalized sequence $(z_n)_{n=1}^\infty$ such that

$$\|\lambda T^k z_n - T^{k+1} z_n\| = \|\lambda z_n - T z_n\| \to 0 \quad (\forall k \in \mathbb{N})\,. \tag{1.48}$$

By (1.47), for each $n \in \mathbb{N}$, there exist k_n and $a_n \in K$ such that

$$\|T^{k_n} z_n - a_n\| < n^{-1}.$$

Passing to a subsequence, one can assume $\|a_n - a\| \to 0$ and $\|T^{k_n} z_n - a\| \to 0$, i.e.,

$$\|T^{k_n} z_n - a\| \to 0\,. \tag{1.49}$$

Combining (1.48) with (1.49), we obtain $Ta = \lambda a$ and $\|a\| = 1$. The obtained contradiction shows that $\dim(X) < \infty$ in the complex case.

II. The real case: Let X be a real Banach space. Consider the complexification: $T_{\mathbb{C}} : X_{\mathbb{C}} \to X_{\mathbb{C}}$, $T_{\mathbb{C}}(x + iy) = Tx + iTy$. The operator $T_{\mathbb{C}}$ is a surective isometry by Exercise 1.1.43, therefore $\sigma(T_{\mathbb{C}}) \subseteq \Gamma$. Since $\dim(X_{\mathbb{C}}) = \infty$, the essential spectrum $\sigma_{ess}(T_{\mathbb{C}}) \subseteq \sigma(T_{\mathbb{C}})$ (see, for example, [2, Def. 7.39, p. 299]) is not empty.

Let $\lambda \in \sigma_{ess}(T_{\mathbb{C}})$, then $\dim(\ker(\lambda I - T_{\mathbb{C}})) = \infty$ or $(\lambda I - T_{\mathbb{C}})(X)$ is not closed in $X_{\mathbb{C}}$. Indeed, assume that $\dim(\ker(\lambda I - T_{\mathbb{C}})) < \infty$ and $(\lambda I - T_{\mathbb{C}})(X)$ is closed. Take a closed complement Y to $\ker(\lambda I - T_{\mathbb{C}})$ in $X_{\mathbb{C}}$ and define an operator $R \in \mathcal{L}(X_{\mathbb{C}})$ by

$$Rx = \begin{cases} x & x \in \ker(\lambda I - T_{\mathbb{C}}) \\ 0 & x \in Y \end{cases}\,.$$

Then $\ker(\lambda I - (T_{\mathbb{C}} + R)) = \{0\}$. Since R is a compact operator,

$$\sigma_{ess}(T_{\mathbb{C}} + R) = \sigma_{ess}(T_{\mathbb{C}}).$$

Thus λ is not an eigenvalue of $T_{\mathbb{C}} + R$ and λ belongs to the boundary of $\sigma(T_{\mathbb{C}} + R)$. So, λ is in the approximative spectrum of $T_{\mathbb{C}} + R$ and, therefore, there exists a normalized sequence $(w_n)_{n=1}^{\infty}$ in $X_{\mathbb{C}}$ such that

$$\lim_{n\to\infty} \|(T_{\mathbb{C}} + R)w_n - \lambda w_n\| = 0. \tag{1.50}$$

The subspaces $(\lambda I - T_{\mathbb{C}})(X)$ and $(\lambda I - (T_{\mathbb{C}} + R))(X)$ are closed or not simultaneously. If they are closed, by the open mapping theorem,

$$T_{\mathbb{C}} + R : \quad X \to (\lambda I - (T_{\mathbb{C}} + R))(X)$$

is an open map, which contradicts the condition (1.50). This shows that

$$\dim(\ker(\lambda I - T_{\mathbb{C}})) = \infty$$

or $(\lambda I - T_{\mathbb{C}})(X)$ is not closed. In both cases, there exists a normalized sequence $z_n = x_n + iy_n \in X_{\mathbb{C}}$ satisfying

$$\|\lambda T_{\mathbb{C}}^k z_n - T_{\mathbb{C}}^{k+1} z_n\| = \|\lambda z_n - T_{\mathbb{C}} z_n\| \to 0 \quad (\forall k \in \mathbb{N}), \tag{1.51}$$

and having no cluster points in $X_{\mathbb{C}}$. Consider the polynomial

$$S_\lambda(t) = (\overline{\lambda} - t)(\lambda - t) = t^2 - t(\lambda + \overline{\lambda}) + |\lambda|^2.$$

The coefficients of $S_\lambda(t)$ are real, so $S_\lambda(T_{\mathbb{C}}) = (S_\lambda(T))_{\mathbb{C}}$. By (1.51),

$$\|S_\lambda(T_{\mathbb{C}}) z_n\| \to 0$$

and, therefore,

$$\lim_{n\to\infty} \|S_\lambda(T) x_n\| = 0 \;\; \& \;\; \lim_{n\to\infty} \|S_\lambda(T) y_n\| = 0\,.$$

If there is a converging subsequence $(x_{n_k})_{k=1}^{\infty}$, then the corresponding subsequence $(y_{n_k})_{k=1}^{\infty}$ has no cluster points. In any case, there exists a normalized sequence $(u_n)_{n=1}^{\infty}$ in X having no cluster points and such that

$$\begin{aligned} \|T^2(T^m u_n) - (\lambda + \overline{\lambda})T(T^m u_n) + |\lambda|^2 T^m u_n\| &= \|T^m \circ S_\lambda(T) u_n\| \\ &= \|S_\lambda(T) u_n\| \\ &\to 0 \end{aligned}$$

for every m. By (1.47), there is $a \in K$ such that

$$T^2 a - (\lambda + \overline{\lambda})Ta + |\lambda|^2 a = 0.$$

Thus the orbit $\{T^n a\}_{n=0}^{\infty}$ of a belongs to the two-dimensional subspace

$$\operatorname{span}\{T^n a \mid n \in \mathbb{Z}\}$$

attracting some subsequence of $(u_n)_{n=1}^{\infty}$. This contradicts the fact that $(u_n)_{n=1}^{\infty}$ has no cluster points. The lemma is proved. □

Remark that this lemma can be proved more directly, by using the fact that, for any λ in $\sigma_{ess}(T)$ belonging to the boundary of an unbounded component of the resolvent set of T, $\dim\ker(\lambda I - T) < \infty$ or the range of $\lambda I - T$ is not closed [60, Ch.IV,5.6].

Proof of Theorem 1.3.29. It is enough to show that T is almost periodic. Indeed, in this case, the set $\{T^n x : n \in \mathbb{N}, x \in K\}$ is a precompact constrictor for T.

Let $x \in B_X$. The orbit $\{T^n x\}_{n=1}^{\infty}$ has a cluster point in K, say a, which is a coming back vector for T (see (1.10)). Consider the following subspace of X:

$$A = \operatorname{span}\{a, Ta, T^2 a, \ldots\}\,.$$

Then, by the same argument as in the proof of Theorem 1.1.26, any vector in the norm closure $\operatorname{cl} A$ of A is a coming back point. Therefore, the restriction $T|_{\operatorname{cl} A}$ of T onto $\operatorname{cl} A$ is a surjective isometry. Lemma 1.3.30 shows that

$$\dim(\operatorname{cl} A) < \infty\,.$$

Thus, the set $\{T^n a\}_{n=1}^{\infty}$ is precompact and, therefore, the set $\{T^n x\}_{n=1}^{\infty}$ is precompact. Since $x \in B_X$ is arbitrary, we obtain that T is almost periodic, and the proof is completed. □

Corollary 1.3.31 (Ansari–Bourdon–Storozhuk). *Let X be a real or complex Banach space,* $\dim(X) > 2$*, and let $T \in \mathcal{L}(X)$ be a supercyclic power bounded operator. Then* $\lim\limits_{n\to\infty} \|T^n x\| = 0$ *for all $x \in X$.*

Proof. Rescaling X by an appropriate equivalent norm, we may suppose $\|T\| \le 1$. Assume $\|T^n a\| \not\to 0$ for some $a \in X$. Since T is supercyclic,

$$\operatorname{span}\{T^n a : n \ge 0\}$$

is dense in X and a is a coming back vector by the same argument as in the proof of Theorem 1.1.26. Therefore, T is a surjective isometry. For any $x \in B_X$, there exist λ_k, $|\lambda_k| \le 1$, and n_k such that

$$\lim_{k\to\infty} \|\lambda_k T^{n_k} a - x\| = 0$$

or, equivalently,

$$\|\lambda_k a - T^{-n_k} x\| \to 0.$$

So, the set

$$K = \{\lambda a : \lambda \in \mathbb{C}, |\lambda| \leq 1\}$$

satisfies (1.47) for the surjective isometry T^{-1}. Lemma 1.3.30 shows that

$$\dim(X) < \infty\,.$$

Thus there exists a non-trivial T-invariant subspace in X (see [2, Cor.10.6,10.7]) which contradicts the supercyclicity of T. Hence, $\|T^n a\| \to 0$. □

Exercise 1.3.32. Let X be a real or complex Banach space, $c \in X$, $\|c\| = 1$, and let $T \in \mathcal{L}(X)$ be a power bounded operator such that

$$\lim_{n\to\infty}\inf \operatorname{dist}(T^n x, \operatorname{span}\{c\}) = 0 \qquad (\forall x \in B_X)\,.$$

(a) Prove that T is constrictive with a compact constrictor

$$\{\lambda \cdot c :\ |\lambda| \leq \sup_{n\geq 1} \|T^n\|\}.$$

Hint: Use Theorem 1.1.24 and arguments such as above in the proof of Theorem 1.3.29.

(b) Formulate and prove an analogue of the previous result for a C_0-semigroup and, more generally, for any bounded operator representation of any abelian semigroup.

(c) Formulate and prove an analogue of Theorem 1.3.29 for a C_0-semigroup and, more generally, for any bounded operator representation of any abelian semigroup.

The following question arises in connection with Theorem 1.3.29.

Open Problem 1.3.33. *Given a bounded one-parameter operator semigroup* $\mathcal{T} = (T_t)_{t\in J}$ *in* X*, a compact set* $C \subseteq X$*, and* $\eta \in \mathbb{R}$*,* $0 \leq \eta < 1$*, such that*

$$\lim_{t\to\infty}\inf \operatorname{dist}(T_t x, C) \leq \eta \qquad (\forall x \in B_X)\,.$$

Is $\mathcal{T}$ *quasi-constrictive?*

1.3.12 There are interesting relations between mean ergodicity and constrictivity of a one-parameter semigroup, discussed in [35] and [38]. In particular, the following two theorems are taken from there, and we refer for the proofs to the above mentioned papers.

Theorem 1.3.34. *Let* X *be a Banach space and let* $T \in \mathcal{L}(X)$ *be such that* $\overline{\lambda}T$ *is mean ergodic for all* $\lambda \in \sigma_\pi(T)$*. Then the following conditions are equivalent:*

(i) *T is quasi-constrictive;*

(ii) *T is constrictive and power bounded.* □

Theorem 1.3.35. *Let* $\mathcal{T} = (T_t)_{t\geq 0}$ *be a bounded* C_0*-semigroup with the generator* G*. Assume that the* C_0*-semigroup* $(e^{-\lambda t}T_t)_{t\geq 0}$ *is mean ergodic for every eigenvalue* λ *in* $\sigma(G) \cap i\,\mathbb{R}$*. Moreover, assume that* T_{t_0} *is quasi-constrictive for some* t_0*. Then* $\mathcal{T}$ *is constrictive.* □

1.3.13 Let $\mathcal{T} = (T_t)_{t\in J}$ be a one-parameter semigroup in a Banach space X.

Exercise 1.3.36. Show that if $\mathcal{T}$ is bounded, then

$$X_0(\mathcal{T}) = \{x \in X : \lim_{t\to\infty} \|T_t x\| = 0\}$$

is a closed subspace of X.

It was pointed out by Storozhuk [122] that, according to the Saxon–Levin result [109], if the subspace $X_0(\mathcal{T})$ is of a countable co-dimension, then $X_0(\mathcal{T})$ is barreled. Hence $\mathcal{T}$ is bounded in $X_0(\mathcal{T})$ and, therefore, $X_0(\mathcal{T})$ is closed in X. Moreover, in this case, the co-dimension of $X_0(\mathcal{T})$ is finite.

Exercise 1.3.37. Show that, for a bounded semigroup $\mathcal{T} = (T_t)_{t\in J}$ in a Banach space X,

$$\lim_{t\to\infty} \operatorname{dist}(T_t x, X_0(\mathcal{T})) = 0 \quad \Rightarrow \quad \lim_{t\to\infty} \|T_t x\| = 0 \qquad (\forall x \in X). \tag{1.52}$$

However, for unbounded semigroups, the condition (1.52) may fail even if the subspace X_0 is closed. To show this, take a sequence $(\beta_j)_{j=1}^{\infty} \subseteq \mathbb{C}$ satisfying

$$|\beta_j| \le 1, \qquad \beta_1 = 1, \qquad \sum_{k=1}^{\infty} \Big| \prod_{j=1}^{k} \beta_j \Big|^2 = \infty, \qquad \textit{and} \qquad \prod_{j=1}^{\infty} \beta_j = 0$$

(for instance, one can take $\beta_1 := 1$ and $\beta_j := \sqrt{\frac{j-1}{j}}$ for $j > 1$). Put

$$\xi_n := \Big(\sum_{k=1}^{n} \Big| \prod_{j=1}^{k} \beta_j \Big|^2 \Big)^{-\frac{1}{3}}$$

for all $n \in \mathbb{N}$, and $\gamma_i := \frac{\xi_i}{\xi_{i-1}}$ for $i > 1$. Remark that

$$0 \le \gamma_j \le \gamma_{j+1} \le 1$$

for all $j > 1$.

Let $\mathcal{H}$ be a separable Hilbert space. Define an operator $T \in \mathcal{L}(\mathcal{H})$ as follows:

$$T(e_{i,j}) = \begin{cases} \gamma_{i+1} e_{i+1,1} + \beta_2 e_{i,2} & j = 1 \\ \beta_{j+1} e_{i,j+1} & j > 1 \end{cases},$$

where $\{e_{i,j}\}_{i,j=1}^{\infty}$ is an orthonormal basis of H. Then

$$\begin{aligned}
\|T^n(e_{i,1})\|^2 &= |\beta_2\beta_3\dots\beta_{n+1}|^2 + |\gamma_{i+1}\beta_2\dots\beta_n|^2 + \dots + |\gamma_{i+1}\gamma_{i+2}\dots\gamma_{i+n}|^2 \\
&\geq \Big(\prod_{l=i+1}^{n+i}\gamma_l\Big)^2 \sum_{k=1}^{n+1}\Big|\prod_{j=1}^{k}\beta_j\Big|^2 \\
&\geq \Big(\prod_{l=2}^{n+1}\gamma_l\Big)^2 \sum_{k=1}^{n+1}\Big|\prod_{j=1}^{k}\beta_j\Big|^2 \\
&= \xi_{n+1}^2 \sum_{k=1}^{n+1}\Big|\prod_{j=1}^{k}\beta_j\Big|^2 \\
&= \Big(\sum_{k=1}^{n+1}\Big|\prod_{j=1}^{k}\beta_j\Big|^2\Big)^{\frac{1}{3}} \\
&\to \infty,
\end{aligned}$$

as $n \to \infty$ for every $i \in \mathbb{N}$. Obviously,

$$\|T^n x\| \geq |x_{i,1}| \cdot \|T^n(e_{i,1})\|$$

for all $x = \sum_{i,j=1}^{\infty} x_{i,j}e_{i,j} \in \mathcal{H}$ and $i \in \mathbb{N}$. Thus, we obtain that

$$\lim_{n\to\infty} \|T^n x\| = \infty$$

whenever $x_{i,1} \neq 0$ for some $i \in \mathbb{N}$. On the other hand, $\prod_{j=1}^{\infty} \beta_j = 0$ and $\beta_j > 0$ imply

$$(\forall x \in \mathcal{H}) \Big[[(\forall i \in \mathbb{N})[x_{i,1} = 0]] \Rightarrow \lim_{n\to\infty} \|T^n x\| = 0\Big].$$

Then

$$\begin{aligned}
\mathcal{H}_0 := \mathcal{H}_0(T) &= \{x \in \mathcal{H} : \lim_{n\to\infty} \|T^n x\| = 0\} \\
&= \{x \in \mathcal{H} : (\forall i \in \mathbb{N})[x_{i,1} = 0]\}
\end{aligned}$$

is a closed subspace of $\mathcal{H}$. At the same time,

$$\begin{aligned}
\lim_{n\to\infty} \operatorname{dist}(T^n x, \mathcal{H}_0) &= \Big(\sum_{p=1}^{\infty} \lim_{n\to\infty} \Big(\prod_{i=p+1}^{p+n} \gamma_i |x_{p,1}|\Big)^2\Big)^{\frac{1}{2}} \\
&= \Big(\sum_{p=1}^{\infty} \lim_{n\to\infty} \frac{\xi_{p+n}^2}{\xi_p^2} |x_{p,1}|^2\Big)^{\frac{1}{2}} \\
&= 0
\end{aligned}$$

holds for all $x \in \mathcal{H}$.

1.3.14 The notion of quasi-constrictive semigroups was introduced in [35] for the discrete case, and in [38] for general abelian semigroups.

Definition 1.3.38. A representation of an abelian semigroup $\mathbb{P}$ in $\mathcal{L}(X)$ is called *asymptotically bounded* if

$$\overline{\lim_{t\to\infty}}\|T_t x\| < \infty \qquad (\forall x \in X).$$

Definition 1.3.39. A representation $\mathcal{T} = (T_t)_{t\in\mathbb{P}}$ of an abelian semigroup $\mathbb{P}$ in $\mathcal{L}(X)$ is called *quasi-constrictive* if $X_0(\mathcal{T})$ is closed and $\mathrm{codim} X_0(\mathcal{T}) < \infty$.

Example 1.3.40. Let $\mathbb{P}$ be the abelian semigroup of all co-finite sets of $\mathbb{N}$ with the operation $A \circ B := A \cap B$, and let H be a separable Hilbert space with the orthonormal basis $\{e_i\}_{i=1}^{\infty}$. Let $P_A \in \mathcal{L}(H)$ be the orthogonal projection onto the closure of $\mathrm{span}\{e_i : i \in A\}$ for all $A \in \mathbb{P}$. Then the representation $\mathcal{P} = (P_A)_{A\in\mathbb{P}}$ is quasi-constrictive, because of

$$H_0(\mathcal{P}) = \{x \in H : \lim_{A\to\infty} \|P_A x\| = 0\} = H\,.$$

At the same time, the semigroup $(P_A^n)_{n=0}^{\infty}$ is not quasi-constrictive for any $A \in \mathbb{P}$.

Example 1.3.41. It is easy to see that a bounded abelian semigroup $\mathcal{G} \subseteq \mathcal{L}(X)$ is quasi-constrictive if the semigroup $(G^n)_{n=0}^{\infty}$ is quasi-constrictive for some $G \in \mathcal{G}$, and Example 1.3.40 shows that the converse is not true.

Let $\mathcal{Q} \subseteq \mathcal{L}(X)$ be a bounded abelian semigroup.

Assume that $\mathrm{codim}\overline{(I-Q)(X)} < \infty$ holds for some $Q \in \mathcal{Q}$. Take the convex hull $\mathcal{G} := \mathrm{co}(\mathcal{Q}, I)$. Then $\mathcal{G} \subseteq \mathcal{L}(X)$ is a bounded abelian semigroup. Consider the operator

$$G := (I+Q)/2 \in \mathcal{G}\,.$$

Since $\sigma(G) \cap \Gamma \subseteq \{1\}$ and G is power bounded, Theorem 1.1.22 implies

$$\begin{aligned} \|G^n \circ (I-Q)\| &= 2\|G^n \circ (G-I)\| \\ &= 2\|G^{n+1} - G^n\| \\ &\to 0. \end{aligned}$$

So

$$\overline{(I-Q)(X)} \subseteq X_0(G)$$

and $G \in \mathcal{G}$ is quasi-constrictive. Then, by the remark above, $\mathcal{G}$ is quasi-constrictive as well.

Example 1.3.42. Let $\mathcal{D}$ be the convolution semigroup of all $f \geq 0$ of norm 1 in $L^1(\mathbb{R}_+)$, and let $\mathcal{S} = (S_t)_{t\geq 0}$ be a bounded C_0-semigroup in a Banach space X. Set

$$C_f = \int_0^{\infty} f(t) S_t \, dt \in \mathcal{L}(Y) \qquad (f \in \mathcal{D}).$$

Then $f \mapsto C_f$ is an abelian semigroup representation of $\mathcal{D}$ in $\mathcal{L}(X)$. It is quasi-constrictive whenever $\mathcal{S}$ is quasi-constrictive. Show that $\mathcal{S}$ is quasi-constrictive. By Example 1.3.41, it is enough to prove existence of a single quasi-constrictive operator C_f. For $f(t) = e^{-t}$, we obtain $C_f = (\lambda - A)^{-1}$, where A is the generator of $\mathcal{S}$. Then

$$C_f^n x = \int_0^\infty e^{-t} \frac{t^n}{n!} S_t x \, dt \,.$$

This in turn yields that C_f is quasi-constricted with

$$X_0(\mathcal{S}) \subseteq X_0(C_f) \,.$$

Let $x \in X_0(\mathcal{S})$ be arbitrary. Then, given $\varepsilon > 0$, there exists t_0 such that $\|S_t x\| < \varepsilon$ for all $t \geq t_0$. So if $M = \sup_{t \geq 0} \|S_t\|$, then

$$\begin{aligned} \|C_f^n x\| &\leq M\|x\| \int_0^{t_0} e^{-t} \frac{t^n}{n!} \, dt + \varepsilon \int_{t_0}^\infty e^{-t} \frac{t^n}{n!} \, dt \\ &\leq M\|x\| \int_0^{t_0} e^{-t} \frac{t^n}{n!} \, dt + \varepsilon \,. \end{aligned}$$

Since the sequence $(\frac{t^n}{n!})_{n=0}^\infty$ converges uniformly to 0 on $[0, t_0]$, we have that $\lim\limits_{n \to \infty} \|C_f^n x\| = 0$ for all $x \in X_0(\mathcal{S})$, and

$$\operatorname{codim}(X_0(C_f)) \leq \operatorname{codim}(X_0(\mathcal{S})) < \infty \,.$$

1.3.15 We reproduce two theorems from [38] without proofs and send the reader to this paper for details.

Theorem 1.3.43. *Let X be a Banach space and let $\mathcal{T} = (T_t)_{t \in \mathbb{P}}$ be an asymptotically bounded representation of an abelian semigroup $\mathbb{P}$ in $\mathcal{L}(X)$. Then the following conditions are equivalent*:

(i) *$\mathcal{T}$ is quasi-constrictive*;

(ii) *for every $\varepsilon > 0$, there exist an equivalent norm $\|\cdot\|_\varepsilon$ on X and $A_\varepsilon \in$* $\mathrm{Constr}_{\|\cdot\|_\varepsilon}(\mathcal{T})$ *such that $\chi_{\|\cdot\|_\varepsilon}(A_\varepsilon) \leq \varepsilon$;*

(iii) *there exist an equivalent norm $\|\cdot\|_1$ on X and $A \in$* $\mathrm{Constr}_{\|\cdot\|_1}(\mathcal{T})$ *such that $\chi_{\|\cdot\|_1}(A) < 1$.* □

We call $\mathcal{T}$ *asymptotically weakly almost periodic*, whenever $\lim_{\mathcal{U}} T_t x$ exists in the weak topology for every $x \in X$, and every ultra-filter $\mathcal{U}$ on $\mathbb{P}$ is finer than the filter $\mathcal{F}$ introduced in **1.3.9** (see also [81] for an equivalent definition). It is clear that asymptotically weakly almost periodic representations are asymptotically bounded, and that in the one-parameter case the asymptotically weakly almost periodicity of a semigroup coincides with the weakly almost periodicity.

Theorem 1.3.44. *Let X be a Banach space and $\mathcal{T} = (T_t)_{t\in\mathbb{P}}$ be an asymptotically bounded representation of an abelian semigroup $\mathbb{P}$ in $\mathcal{L}(X)$. Then the following conditions are equivalent*:

(i) *$\mathcal{T}$ is asymptotically weakly almost periodic and quasi-constrictive*;

(ii) *$\mathcal{T}$ is constrictive*;

(iii) *there exist $A \in \mathrm{Constr}(\mathcal{T})$ such that $\chi(A) = 0$*;

(iv) *$\mathcal{T}$ is asymptotically weakly almost periodic and, for some equivalent norm $\|\cdot\|_1$ on X, there exists $A \in \mathrm{Constr}_{\|\cdot\|_1}(\mathcal{T})$ such that $\chi_{\|\cdot\|_1}(A) < 1$.* □

1.3.16 We finish this section with a short discussion of ultra-powers of Banach spaces which were used in this section and will be used later in Sections 2.2 and 3.2. For a more detailed explanation we refer to [2], [85], [110], and [112]. Let Γ be a non-empty set and let $\mathcal{U} \subseteq \mathcal{P}(\Gamma)$ be a free ultra-filter. Denote by $\mu_{\mathcal{U}}$ the finitely additive $\{0,1\}$-valued measure on Γ, given by $\mu_{\mathcal{U}}(A) = 1$ iff $A \in \mathcal{U}$. Note that the intersection of finitely many sets of $\mu_{\mathcal{U}}$-measure one has also measure $\mu_{\mathcal{U}}$-measure one.

We need the following construction. Let $(X_\gamma)_{\gamma\in\Gamma}$ be a family of Banach spaces. Denote by X_∞ the Banach space of all *$\mathcal{U}$-bounded* functions $\tilde{x} = (x_\gamma)_{\gamma\in\Gamma}$ on Γ such that $x_\gamma \in X_\gamma$ for all $\gamma \in \Gamma$, equipped with the semi-norm

$$\|\tilde{x}\| = \inf\{M \in \mathbb{R}_+ : \mu_{\mathcal{U}}(\{\gamma \in \Gamma : \|x_\gamma\| \le M\}) = 1\}\,.$$

Then

$$X_0 := \ker(\|\cdot\|) = \{\tilde{x} = (x_\gamma) : \lim_{\mathcal{U}} \|x_\gamma\| = 0\}$$

is a closed subspace of X_∞, where we assume under $\lim_{\mathcal{U}} \|x_\gamma\| = 0$ the following property:

$$(\forall\varepsilon > 0) \quad \mu_{\mathcal{U}}(\{\gamma \in \Gamma : \|x_\gamma\|_{X_\gamma} \le \varepsilon\}) = 1\,.$$

The quotient $\hat{X} := X_\infty/X_0$ is called the (bounded) *ultra-product* of the family $(X_\gamma)_{\gamma\in\Gamma}$ with respect to $\mathcal{U}$. The norm on $\hat{X}$ is given by

$$\|\hat{y}\| = \|\tilde{y} + X_0\| = \lim_{\mathcal{U}} \|y_\gamma\|\,. \tag{1.53}$$

Remark that the limit in (1.53) exists since the family $\|y_\gamma\|$ of reals is $\mathcal{U}$-bounded.

If all X_γ are equal to some single Banach space E, we shall denote

$$\ell^\infty_{\mathcal{U}}(E) := E_\infty \quad \text{and} \quad c_{\mathcal{U}}(E) := E_0\,.$$

Then $E_{\mathcal{U}} = \hat{E} := \ell^\infty_{\mathcal{U}}(E)/c_{\mathcal{U}}(E)$ is called the (bounded) *ultra-power* of E with respect to $\mathcal{U}$. In this case, there is a natural isometric embedding of E into $\hat{E}$ by means of $x \to (x)_\gamma + c_{\mathcal{U}}(E)$. Given a family $\{x_\gamma\}_{\gamma\in\Gamma} \in \ell^\infty_{\mathcal{U}}(E)$ and a vector $x \in E$, we say that $x = \|\cdot\|{-}\lim_{\mathcal{U}} x_\gamma$, the *norm-limit with respect to* $\mathcal{U}$, if $\lim_{\mathcal{U}} \|x_\gamma - x\| = 0$.

Remark that every norm-precompact family $\{x_\gamma\}_{\gamma\in\Gamma}$ in E possesses the unique norm-limit with respect to $\mathcal{U}$. A similar definition is for $w-\lim_{\mathcal{U}} x_\gamma$, namely, the *weak-limit with respect to* $\mathcal{U}$. Every weakly precompact family $\{x_\gamma\}_{\gamma\in\Gamma}$ in E possesses the unique weak-limit with respect to $\mathcal{U}$. See the Abramovich–Aliprantis book [2, Lm.1.59] for details.

1.3.17 Every bounded family $(S_\gamma)_{\gamma\in\Gamma}$ of operators S_γ in X_γ defines the operator $\tilde{S}$ in X_∞ by $\tilde{S}\tilde{x} = (S_\gamma x_\gamma)_{\gamma\in\Gamma}$, the norm of which is given by $\|\tilde{S}\| = \sup_{\gamma\in\Gamma}\|S_\gamma\|$. So X_0 is invariant for $\tilde{S}$, and we obtain the uniquely defined operator $\hat{S}$ in $\hat{X}$ given by $\hat{S}\hat{y} := \tilde{S}\tilde{y} + X_0$. Moreover, $\|\hat{S}\| = \lim_{\mathcal{U}}\|S_\gamma\|$. If $(R_\gamma)_{\gamma\in\Gamma}$ is another bounded family of operators R_γ in X_γ such that

$$\mu_{\mathcal{U}}(\{\gamma : R_\gamma = S_\gamma\} = 1,$$

then $\hat{R} = \hat{S}$. Thus each subfamily $(S_\gamma)_{\gamma\in M}$ with $\mu_{\mathcal{U}}(M) = 1$ defines in a canonical way an operator on $\hat{X}$ which coincides with $\hat{S}$. It should cause no confusion if we denote the operator on $\hat{X}$ induced by $(S_\gamma)_{\gamma\in M}$ also by $\hat{S}$.

If the spaces X_γ are all Banach lattices, so is $\hat{X}$, and if $(S_\gamma)_{\gamma\in\Gamma}$ consists of positive operators, then $\hat{S}$ is positive. If all X_γ are equal to some Banach space X, then each $T\in\mathcal{L}(X)$ has a canonical extension $\hat{T}$ on $\hat{X}$ given by

$$\hat{T}\hat{y} = (Ty_\gamma)_{\gamma\in\Gamma} + c_{\mathcal{U}}(X)\,.$$

$\hat{T}$ is called an *ultra-power* of T.

We give the following theorem about spectral properties of $\hat{T}$. For the proof, we refer to Schaefer [110] or Meyer-Nieberg [85].

Theorem 1.3.45. *Let T be a bounded operator in a Banach space X, then $\sigma(\hat{T}) = \sigma(T)$. Moreover, every point in the approximative spectrum of T is an eigenvalue of $\hat{T}$.* □

Chapter 2

Positive semigroups in ordered Banach spaces

In this chapter, we deal with one-parameter positive semigroups in ordered Banach spaces. Firstly, we discuss the notion of ideally ordered Banach spaces and uniformly order convex Banach spaces. Both classes include L^p-spaces $(1 \leq p < \infty)$ as well as preduals of von Neumann algebras. We prove several theorems about positive semigroups in such Banach spaces. Then we consider positive semigroups in Banach lattices and investigate several types of asymptotic regularity of these semigroups. In the last section of this chapter, we deal with relations between the geometry of Banach lattices and mean ergodicity of bounded positive semigroups in them.

2.1 Ordered Banach spaces, constrictors, domination

Here we recall basic properties of ordered Banach spaces and positive operators in them. For the general theory of ordered Banach spaces, we refer to [17], [112], and [129]. Then we introduce and discuss two important classes of spaces: the ideally ordered Banach spaces and uniformly order convex Banach spaces. We state Theorem 2.1.8 which is a basis for several results of Sections 3.2 and 3.3. Then we present the notion of asymptotic domination and study it for positive semigroups in ideally ordered Banach spaces. We give some properties of powers of mean ergodic operators in ideally ordered Banach spaces, the proofs of which are closely related to the proof of one of the main results of this section, namely Theorem 2.1.11 concerning asymptotic domination. Then we investigate the strong stability and almost periodicity of asymptotically dominated positive semigroups in uniformly order convex Banach spaces.

2.1.1 Let X be a real Banach space. A subset X_+ is called a *positive cone* in X if it is closed and satisfies the following properties:

$$\mathbb{R}_+ \cdot X_+ + \mathbb{R}_+ \cdot X_+ \subseteq X_+; \quad -X_+ \cap X_+ = \{0\}; \quad X_+ - X_+ = X.$$

The elements of X_+ are called *positive*. The pair (X, X_+) is called an *ordered Banach space* (if a positive cone X_+ in X is fixed, we call (X, X_+) simply X). The ordering "$\le$" on X is introduced as follows:

$$x \le y \;\text{ if and only if }\; y - x \in X_+ \,.$$

For $x \le y$ in X, denote by $[x, y]$ the *order interval*

$$\{z \in X : x \le z \le y\}.$$

Recall that a positive element u of an ordered Banach space X is called an *order unit* if the interval $[0, u]$ has a non-empty interior. A positive element w of X is called a *weak order unit* if any non-trivial order interval $[0, x]$ has a non-trivial intersection with the interval $[0, w]$.

Definition 2.1.1. A subset A of X is called *order bounded* if $A \subseteq [-z, z]$ for some $z \in X_+$. A is called *almost order bounded* if for any $\varepsilon > 0$ there exists $z_\varepsilon \in X_+$ such that

$$A \subseteq [-z_\varepsilon, z_\varepsilon] + \varepsilon \cdot B_X \,.$$

A is called *quasi-order bounded* if there exist $z \in X_+$ and $0 \le \eta < 1$ such that

$$A \subseteq [-z, z] + \eta \cdot B_X \,.$$

The positive cone X_+ in an ordered Banach space X is called *normal* if all order intervals in X are norm bounded. On any normal ordered Banach space X, there exists an equivalent norm that is *monotone* on X_+, i.e., if $x, y \in X$ such that $0 \le x \le y$, then $\|x\| \le \|y\|$. For example, one can obtain such a monotone norm $\|\cdot\|_1$ in the following way:

$$\|x\|_1 := \sup\{\|y\| : 0 \le y \le x\} \;\text{ if }\; x \ge 0 \;\text{ and}$$

$$\|u\|_1 := \inf\{\|v\|_1 + \|w\|_1 : v, w \ge 0, v - w = u\}$$

for an arbitrary $u \in X_+$. In our book, all ordered Banach spaces are assumed to be equipped with normal positive cones.

Definition 2.1.2. A linear operator T in an ordered Banach space X is called *positive* if $T(X_+) \subseteq X_+$. A semigroup $\mathcal{S} \subseteq \mathcal{L}(X)$ is called *positive* if it consists of positive operators.

The following simple property of positive projections is very useful.

Proposition 2.1.3. *Let X be an ordered Banach space and $P, Q \in \mathcal{L}(X)$ be projections such that $0 \le Q \le P$. Then $Q = Q \circ P \circ Q$. In particular,* $\operatorname{rank}(Q) \le \operatorname{rank}(P)$.

Proof. This follows from

$$\begin{aligned} 0 \le Q \circ P \circ Q - Q &= Q \circ (P - Q) \circ Q \\ &\le P \circ (P - Q) \circ Q \\ &= P \circ Q - P \circ Q \\ &= 0. \end{aligned}$$

□

2.1.2 Now we introduce a class of ordered Banach spaces (cf. [37, Def.1]) that will be intensively investigated below. Let X be an ordered Banach space, then the normality of the positive cone X_+ is equivalent to the norm-continuity of the mapping

$$X_+ \times X_+ \ni (x, y) \to \mathrm{dist}([0, x], [0, y])$$

at the point $(0, 0)$. We define X_+ to be *strongly normal* if this mapping is norm-continuous everywhere on $X_+ \times X_+$. We call X a *strongly normal* Banach space if its positive cone is strongly normal.

Proposition 2.1.4. *If an ordered Banach space X possesses the* interval decomposition property

$$[0, x + y] \;=\; [0, x] \;+\; [0, y] \qquad (\forall x, y \in X_+),$$

then there exists a constant C such that

$$\mathrm{dist}([0, x]\,,\,[0, y]) \le C\|x - y\| \qquad (\forall x, y \in X_+),$$

and, henceforth, X_+ is strongly normal. In particular, any Banach lattice is a strongly normal Banach space.

Proof. By passing to an equivalent norm if necessary, we may assume the norm $\|\cdot\|$ to be monotone on X_+. Let $0 \le x, y \in X$ be given. Then $x - y = q - r$, where $q, r \ge 0$ and $\|q\|\,,\,\|r\| \le C\|x - y\|$. Here we use the normality of the positive cone X_+. Therefore, if $0 \le u \le x$, then

$$0 \le u \le y + q\,,$$

hence $u = u_1 + u_2$, where $0 \le u_1 \le y$ and $0 \le u_2 \le q$. This in turn implies

$$\mathrm{dist}(u, [0, y]) \le \|q\| \le C\|x - y\|\,,$$

from which we obtain

$$\mathrm{dist}([0, x]\,,\,[0, y]) \le C\|x - y\|$$

and, finally, $\mathrm{dist}_H([0, x]\,,\,[0, y]) \le C\|x - y\|$. □

2.1.3 We call an ordered Banach space X *ideally ordered* if X_+ is strongly normal and all order intervals in X are weakly compact. We shall intensively use the following simple property of ideally ordered Banach spaces (cf. [37, Thm. 10]).

Theorem 2.1.5. *Let X be an ideally ordered Banach space. If $\{x_n\}_{n=1}^{\infty} \subseteq X_+$ and $x_n \xrightarrow{\|\cdot\|} x_0$, then the set $\bigcup_{n=1}^{\infty} [0, x_n]$ is conditionally weakly compact and its norm-closure contains the order interval $[0, x_0]$. Moreover, if all order intervals $[0, x_n]$ $(n \geq 1)$ are compact, then the set $\bigcup_{n=0}^{\infty} [0, x_n]$ is conditionally compact.*

Proof. Due to the strong normality of X_+,

$$\mathrm{dist}([0, x_n], [0, x_0]) \to 0 \qquad (n \to \infty)\,.$$

Hence, for any $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}$ such that

$$\bigcup_{n=1}^{\infty} [0, x_n] \subseteq \bigcup_{n=0}^{n_\varepsilon} [0, x_n] + \varepsilon B_X. \tag{2.1}$$

Since the set $\bigcup_{n=0}^{n_\varepsilon} [0, x_n]$ is weakly compact for any n_ε, it follows easily that $\bigcup_{n=1}^{\infty} [0, x_n]$ is conditionally weakly compact. Moreover, since

$$\mathrm{dist}([0, x_0], [0, x_n]) \to 0 \qquad (n \to \infty), \tag{2.2}$$

we obtain

$$[0, x_0] \subseteq \mathrm{cl}_{\,\|\cdot\|} \bigcup_{n=1}^{\infty} [0, x_n].$$

Finally, assume that all intervals $[0, x_n]$ are compact. Then the interval $[0, x_0]$ is compact by (2.2). It follows now from (2.1) that the set $\bigcup_{n=0}^{\infty} [0, x_n]$ is conditionally compact. □

As examples of ideally ordered Banach spaces we can take Banach lattices with order continuous norm, for instance, L^p-spaces, where $1 \leq p < \infty$. For another type of examples, we send the reader to Section 3.3.

2.1.4 We shall use the following definition (cf. [32, Def.2.1]) of another class of ordered Banach spaces.

Definition 2.1.6. Let X be an ordered Banach space whose norm $\|\cdot\|$ is monotone on X_+; then X is called uniformly order convex if for every $\varepsilon > 0$ there exists $\delta > 0$ such that, whenever $x, y \in B_X,\ 0 \leq x \leq y$, and $\|y\| - \|x\| < \delta$, then $\|x - y\| < \varepsilon$.

One can easily see that the following equivalence holds.

Lemma 2.1.7. *An ordered Banach space X is uniformly order convex iff for every $M > 0$ and $\varepsilon > 0$ there exists $\delta > 0$ such that if*

$$0 \le x, y \in M \cdot B_X, \quad \operatorname{dist}(y - x, X_+) < \delta, \quad \text{and} \quad \|y\| - \|x\| < \delta,$$

then $\|y - x\| < \varepsilon$. □

Let us mention some examples of uniformly order convex Banach spaces. Every ordered Banach space E with uniformly convex norm is uniformly order convex (cf. [78, Sect.1.e]). In particular, every L^p-space, $1 \le p < \infty$, is uniformly order convex. Moreover, every ordered Banach space E whose norm is additive on E_+ is uniformly order convex. For more examples of these spaces, we can consider so-called non-commutative L^1-spaces (for details, we refer to Section 3.3).

2.1.5 Now we turn back to positive semigroups, and present an important theorem about positive abelian semigroups possessing quasi-order bounded constrictors (see [37, Thm. 6]). Here we assume that J is an abelian semigroup ordered according to **1.3.9**, with a representation $\mathcal{T} = (T_t)_{t \in J}$ in $\mathcal{L}(X)$. The notion of a constrictor for $\mathcal{T}$ is also taken from **1.3.9**.

Theorem 2.1.8 (Emel'yanov–Wolff). *Let X be a strongly normal Banach space, and $\mathcal{T} = (T_t)_{t \in J}$ be a representation of an abelian semigroup J in $\mathcal{L}_+(X)$. If $\mathcal{T}$ has a constrictor*

$$[-y, y] + \eta B_X$$

for some $y \in X_+$ and for some real η, $0 \le \eta < 1$, and if the closure of the convex hull of the orbit $\mathcal{T}y := \{T_t y\}_{t \in J}$ contains a $\mathcal{T}$-invariant point w, then the order interval

$$\frac{1}{1-\eta}[-w, w]$$

is a constrictor of $\mathcal{T}$.

Proof. First of all, $M_{\mathcal{T}} := \sup_{t \in \mathcal{J}} \|T_t\| < \infty$ by the uniform boundedness principle.

Now let $\eta < \sigma < 1$ be fixed. We claim that $\frac{1}{1-\sigma}[-w, w] \in \operatorname{Constr}(\mathcal{T})$. To prove this, it is sufficient to show that for arbitrary $x \in B_X$ and $\varepsilon > 0$ there exists $t(x, \varepsilon) \in J$ satisfying

$$T_t x \in \frac{1}{1-\sigma}[-w, w] + \varepsilon B_X$$

for all $t \ge t(x, \varepsilon)$.

Fix $x \in B_X$ and $\varepsilon > 0$. Since of the cone X_+ is strongly normal, there exists $\delta > 0$ for which

$$\|z - \frac{1}{1-\sigma} w\| \le \delta \;\Rightarrow$$

$$[-z, z] \subseteq \frac{1}{1-\sigma}[-w, w] + \frac{\varepsilon}{2M_{\mathcal{T}}} B_X \qquad (0 \le z \in X). \tag{2.3}$$

Since $w \in \overline{\text{co}}\{T_t y : t \in J\}$, we can take $\alpha_q \in \mathbb{R}_+$ and $s_q \in J$, where $q \in \overline{1,m}$ such that

$$\sum_{q=1}^{m} \alpha_q = 1;\ \ a_m := \sum_{q=1}^{m} \alpha_q T_{s_q} y;\ \ \|a_m - w\| \le \frac{\delta(1-\sigma)}{2M_{\mathcal{T}}}. \tag{2.4}$$

By induction, we construct an increasing sequence $(t_i)_{i=1}^{\infty} \subseteq J$ satisfying

$$T_{t_i - s_q} x \in \big[-\sum_{j=1}^{i} \sigma^{j-1} T_{t_i - t_j} y,\ \sum_{j=1}^{i} \sigma^{j-1} T_{t_i - t_j} y\big] + \sigma^i B_X \tag{2.5}$$

for all $i, q \in \mathbb{N}$, $q \le m$. The first step of (2.5) follows directly from the fact that $[-y, y] + \eta B_X \in \text{Constr}(\mathcal{T})$. Find for some i an element t_i satisfying (2.5) for all $\mathbb{N} \ni q \le m$. Then we choose u_q^i and v_q^i such that

$$u_q^i \in \big[-\sum_{j=1}^{i} \sigma^{j-1} T_{t_i - t_j} y,\ \sum_{j=1}^{i} \sigma^{j-1} T_{t_i - t_j} y\big],$$

$$\|v_q^i\| \le \sigma^i,\ u_q^i + v_q^i = T_{t_i - s_q} x$$

for each q, $\mathbb{N} \ni q \le m$. Then we have, for a large enough t_{i+1},

$$\begin{aligned} T_{t_{i+1} - s_q} x &= T_{t_{i+1} - t_i}(u_q^i + v_q^i) \\ &\in \big[-T_{t_{i+1} - t_i} \sum_{j=1}^{i} \sigma^{j-1} T_{t_i - t_j} y,\ T_{t_{i+1} - t_i} \sum_{j=1}^{i} \sigma^{j-1} T_{t_i - t_j} y\big] \\ &\quad + \sigma^i [-y, y] + \sigma^{i+1} B_X \\ &\subseteq \big[-\sum_{j=1}^{i+1} \sigma^{j-1} T_{t_{i+1} - t_j} y,\ \sum_{j=1}^{i+1} \sigma^{j-1} T_{t_{i+1} - t_j} y\big] + \sigma^{i+1} B_X \end{aligned}$$

for all $q \in \overline{1,m}$. Thus, we obtain that (2.5) is true when replacing i by $i+1$.

Using the fact that w is a fixed point of $\mathcal{T}$ and the condition (2.4), we obtain

$$\begin{aligned} \|\sum_{j=1}^{i} \sigma^{j-1} T_{t_i - t_j} a_m - \frac{1}{1-\sigma} w\| &\le \|\sum_{j=1}^{i} \sigma^{j-1} T_{t_i - t_j} a_m - \frac{1-\sigma^i}{1-\sigma} w\| + \frac{\sigma^i}{1-\sigma}\|w\| \\ &= \|\sum_{j=1}^{i} \sigma^{j-1} T_{t_i - t_j}(a_m - w)\| + \frac{\sigma^i}{1-\sigma}\|w\| \\ &\le \sum_{j=1}^{i} \sigma^{j-1} M \|a_m - w\| + \frac{\sigma^i}{1-\sigma}\|w\| \\ &\le \frac{M}{1-\sigma}\|a_m - w\| + \frac{\sigma^i}{1-\sigma}\|w\| \\ &< \delta/2 + \frac{\sigma^i}{1-\sigma}\|w\| \end{aligned}$$

for all i.

Fix a large enough i such that

$$\sigma^i \leq \min\left(\frac{\delta(1-\sigma)}{2\|w\|}\ ,\ \frac{\varepsilon}{2M_{\mathcal{T}}^2}\right), \tag{2.6}$$

then the condition (2.3) implies

$$\Big[-\sum_{j=1}^{i}\sigma^{j-1}T_{t_i-t_j}a_m,\ \sum_{j=1}^{i}\sigma^{j-1}T_{t_i-t_j}a_m\Big] \subseteq \frac{1}{1-\sigma}[-w,w] + \frac{\varepsilon}{2M_{\mathcal{T}}}B_X. \tag{2.7}$$

Now, by using (2.5), (2.6), and (2.7), we obtain

$$\begin{aligned}
T_{t_i}x &= \sum_{q=1}^{m}\alpha_q T_{s_q+t_i-s_q}x \\
&\in \Big[-\sum_{q=1}^{m}\alpha_q T_{s_q}\sum_{j=1}^{i}\sigma^{j-1}T_{t_i-t_j}y,\ \sum_{q=1}^{m}\alpha_q T_{s_q}\sum_{j=1}^{i}\sigma^{j-1}T_{t_i-t_j}y\Big] \\
&\quad + \sum_{q=1}^{m}\alpha_q T_{s_q}(\sigma^i B_X) \\
&\subseteq \Big[-\sum_{j=1}^{i}\sigma^{j-1}T_{t_i-t_j}a_m,\ \sum_{j=1}^{i}\sigma^{j-1}T_{t_i-t_j}a_m\Big] + \sigma^i M_{\mathcal{T}}B_X \\
&\subseteq \frac{1}{1-\sigma}[-w,w] + \frac{\varepsilon}{2M_{\mathcal{T}}}B_X + \sigma^i M_{\mathcal{T}}B_X \\
&\subseteq \frac{1}{1-\sigma}[-w,w] + \frac{\varepsilon}{M_{\mathcal{T}}}B_X.
\end{aligned}$$

Then

$$\begin{aligned}
T_t x &\in \frac{1}{1-\sigma}[-T_{t-t_i}w, T_{t-t_i}w] + \frac{\varepsilon}{M_{\mathcal{T}}}T_{t-t_i}(B_X) \\
&\subseteq \frac{1}{1-\sigma}[-w,w] + \varepsilon B_X
\end{aligned}$$

for all $J \ni t \geq t(x,\varepsilon) := t_i$.

Thus we have shown that

$$\frac{1}{1-\sigma}[-w,w] \in \mathrm{Constr}(\mathcal{T})\,.$$

By arbitrariness of σ, $\eta < \sigma < 1$, we obtain $\frac{1}{1-\eta}[-w,w] \in \mathrm{Constr}(\mathcal{T})$. □

2.1.6 Our next results concern the inheritance under domination of some asymptotic regularity properties of one-parameter positive semigroups. We need the following definition which presents the generalized form of the domination.

Let X be an ordered Banach space, and let $\mathcal{S} = (S_t)_{t\in J}$, $\mathcal{T} = (T_t)_{t\in J}$ be one-parameter positive semigroups in X, and let $P : X_+ \to X_+$ be an arbitrary mapping.

Definition 2.1.9. We say that

(i) $\mathcal{S}$ is *asymptotically dominated* by $\mathcal{T}$ if

$$\lim_{t\to\infty} \operatorname{dist}(T_t x - S_t x, X_+) = 0 \quad (\forall x \in X_+);$$

(ii) $\mathcal{S}$ is *asymptotically dominated* by P if

$$\lim_{t\to\infty} \operatorname{dist}(Px - S_t x, X_+) = 0 \quad (\forall x \in X_+).$$

We say that a positive operator S is *asymptotically dominated* by a positive operator T if the semigroup $(S^n)_{n=0}^{\infty}$ is asymptotically dominated by $(T^n)_{n=0}^{\infty}$.

The part (i) of Definition 2.1.9 can be given in an equivalent form according to the following proposition.

Proposition 2.1.10. *The following conditions are equivalent*:

(i) *$\mathcal{S}$ is asymptotically dominated by $\mathcal{T}$*;

(ii) *for any $f \in X_+$, there exists a family $(q_t^f)_{t\in J} \subseteq X_+$ such that $\lim_{t\to\infty} \|q_t^f\| = 0$ and $T_t f + q_t^f \geq S_t f$ for all $t \in J$.*

Proof. (i) $\Rightarrow$ (ii): Let $f \in X_+$. Take a family $(h_t)_{t\in J} \subseteq X$ such that

$$T_t f - S_t f - h_t \in X_+ \text{ and } \|h_t\| \to 0.$$

As in the proof of Proposition 2.1.4, take families $(l_t)_{t\in J}$ and $(r_t)_{t\in J}$ such that $h_t = l_t - r_t$, where

$$l_t, r_t \geq 0 \text{ and } \|l_t\|, \|r_t\| \leq C\|h_t\|$$

for all $t \in J$. Then the family $q_t^f := r_t$, satisfies the condition (ii).

(ii) $\Rightarrow$ (i): Let $f \in X_+$. Take a family $(q_t^f)_{t\in J}$, which satisfies (ii). Then $\lim_{n\to\infty} \operatorname{dist}(T^n f - S^n f, X_+) \leq \lim_{n\to\infty} \|q_n^f\| = 0$. □

Remark that the asymptotic domination can be quite far from its special case $0 \leq S \leq T$. For example, a positive semigroup $\mathcal{S} = (S^n)_{n=1}^{\infty}$, which converges strongly to a projection P, is asymptotically dominated by P but, in general, $S \not\leq P$.

2.1.7 The following theorem is about preserving the mean ergodicity under the asymptotic domination.

Theorem 2.1.11. *Let S and T be power bounded positive operators in an ideally ordered Banach space X such that S is asymptotically dominated by T, and T is mean ergodic. Then S is mean ergodic. Moreover, if the mean ergodic projection P_T has finite rank, then the corresponding mean ergodic projection P_S has finite rank and* $\operatorname{rank}(P_S) \leq \operatorname{rank}(P_T)$.

Proof. It is enough to show that $\lim_{n\to\infty} \mathcal{A}_n^S f$ exists for all $f \in X_+$. Fix some $f \in X_+$ and take a sequence $(q_n^f)_{n=1}^\infty \subseteq X_+$ which satisfies the condition (ii) of Proposition 2.1.10. Put $q_0^f := 0$ and $d_n^f := \frac{1}{n}\sum_{k=0}^{n-1} q_k^f$ for all $n \in \mathbb{N}$. Then

$$0 \leq \mathcal{A}_n^S f \leq \mathcal{A}_n^T f + d_n^f \quad (\forall n \in \mathbb{N}).$$

Since T is mean ergodic, $\lim_{n\to\infty} \|\mathcal{A}_n^T f - u\| = 0$ for some $u \in X_+$. Moreover

$$\lim_{n\to\infty} \|\mathcal{A}_n^T f + d_n^f - u\| = 0,$$

since $\lim_{n\to\infty} \|d_n^f\| = 0$. Theorem 2.1.5 implies that the set $\{\mathcal{A}_n^S f\}_{n=1}^\infty$ is conditionally weakly compact and, henceforth, has a weak cluster point. Theorem 1.1.7 implies that $\lim_{n\to\infty} \mathcal{A}_n^S f$ exists. The rest of the proof follows from Proposition 2.1.3. □

For Banach lattices with order continuous norm, this result is due to Arendt and Batty [12]; in the general case, it was stated in [37, Thm.13].

The proof of Theorem 2.1.11 cannot be directly extended to the strongly continuous case. We give the analogue of Theorem 2.1.11 for usual domination only with essentially the same proof.

Theorem 2.1.12. *Let X be an ideally ordered Banach space. Assume that $\mathcal{S} = (S_t)_{t\geq 0}$ is a positive C_0-semigroup in X satisfying $S_t x \leq T_t x$ for all $x \in X_+$, $t \geq 0$, where $\mathcal{T} = (T_t)_{t\geq 0}$ is a bounded mean ergodic C_0-semigroup. Then $\mathcal{S}$ is mean ergodic. Moreover, if the mean ergodic projection $P_\mathcal{T}$ has finite rank, then the corresponding projection $P_\mathcal{S}$ has finite rank and* $\operatorname{rank}(P_\mathcal{S}) \leq \operatorname{rank}(P_\mathcal{T})$.

Proof. Obviously $\mathcal{S}$ is bounded. Show that the net $(\mathcal{A}_t^\mathcal{S} f)_{t\geq 0}$ possesses a weak cluster point for all $f \in X_+$. Fix some $f \in X_+$, then

$$0 \leq \mathcal{A}_t^\mathcal{S} f \leq \mathcal{A}_t^\mathcal{T} f \qquad (\forall t \in \mathbb{R}_+).$$

Since $\mathcal{T}$ is mean ergodic, $\lim_{t\to\infty} \|\mathcal{A}_t^\mathcal{T} f - u\| = 0$ for some $u \in X_+$. Take arbitrarily an increasing sequence $(t_n)_{n=1}^\infty$ such that $\lim_{n\to\infty} t_n = \infty$. Theorem 2.1.5 implies that the set $\{\mathcal{A}_{t_n}^\mathcal{S} f\}_{n=1}^\infty$ is conditionally weakly compact and so has a weak cluster point, that is obviously a weak cluster point of $\{\mathcal{A}_t^\mathcal{S} f : t \geq 0\}$. Theorem 1.1.7 implies that $\lim_{t\to\infty} \mathcal{A}_t^\mathcal{S} f$ exists, which is required. The rest of the proof follows again from Proposition 2.1.3. □

2.1.8 The next result is related to the inheritance of the almost periodicity of a one-parameter positive semigroup under the asymptotic domination.

Theorem 2.1.13. *Let X be an ideally ordered Banach space, and let $\mathcal{S} = (S_t)_{t\in J}$, $\mathcal{T} = (T_t)_{t\in J}$ be one-parameter positive semigroups in X such that $\mathcal{S}$ is asymptotically dominated by $\mathcal{T}$. Assume that $\mathcal{T}$ is almost periodic. Then the semigroup $\mathcal{S}$ is weakly almost periodic. Moreover, if every order interval in X is compact, then $\mathcal{S}$ is almost periodic. Here we have the same as the above inequality of ranks of the corresponding Jacobs–Deleeuw–Glicksberg projections*: $\mathrm{rank}(P_{\mathcal{S}}) \le \mathrm{rank}(P_{\mathcal{T}})$.

Proof. It is enough to show that the sequence $(S_{t_k}x)_{k=1}^{\infty}$ has a weakly convergent subsequence for each $x \in X_+$ and for each strictly increasing sequence $(t_k)_{k=1}^{\infty}$.

Fix $x \in X_+$ and a strictly increasing sequence $(t_k)_{k=1}^{\infty}$. Since $\mathcal{T}$ is almost periodic, there exists a convergence subsequence $(T_{t_{k_i}}x)_{i=1}^{\infty}$. Then for any $\varepsilon > 0$, there exists $i(\varepsilon)$ such that

$$\{S_{t_{k_i}}x\}_{i=i(\varepsilon)}^{\infty} \subseteq \bigcup_{i=i(\varepsilon)}^{\infty} [0, T_{t_{k_i}}x] + \varepsilon \cdot B_X\,. \tag{2.8}$$

By Theorem 2.1.5 and (2.8), the sequence $(S_{t_k}x)_{k=1}^{\infty}$ has a weakly convergent subsequence and $\mathcal{S}$ is weakly almost periodic.

Assume that every order interval in X is compact. The same argument as above shows that the sequence $(S_{t_k}x)_{k=1}^{\infty}$ has a norm-convergent subsequence. So $\mathcal{S}$ is almost periodic. □

2.1.9 Now we present a result (cf. [37, Thm.11]) about powers of a mean ergodic operator. According to Theorem 1.1.12, any power bounded operator T in a Banach space X is mean ergodic, provided that T^m is mean ergodic for some m. In general, the converse is not true, even for *Koopman operators* in $C(K)$ [118]. But it is true for many important classes of operators.

Theorem 2.1.14. *Let X be an ideally ordered Banach space, and let T be a positive power bounded mean ergodic operator in X. Then T^m is mean ergodic for all $m \in \mathbb{N}$.*

Proof. Let $m \in \mathbb{N}$. It is enough to show that $\lim\limits_{n\to\infty} \mathcal{A}_n^{T^m} f$ exists for all $f \in X_+$. Fix some $f \in X_+$. Then

$$0 \le \mathcal{A}_n^{T^m} f = \frac{1}{n}\sum_{k=0}^{n-1} T^{mk} f \le \frac{1}{n}\sum_{i=0}^{mn-1} T^i f = m\mathcal{A}_{mn}^{T} f.$$

Since T is mean ergodic, the sequence $(m\mathcal{A}_{mn}^{T} f)_{n=1}^{\infty}$ is convergent in X. By Theorem 2.1.5, the set

$$\{A_n^{T^m} f\}_{n=1}^{\infty}$$

is conditionally weakly compact and, consequently, possesses a weak cluster point. Applying Theorem 1.1.7, we obtain that $\lim\limits_{n\to\infty} \mathcal{A}_n^{T^m} f$ exists. □

Remark that the conditions of the theorem are satisfied for Banach lattices with order continuous norm, as well as for self-adjoint parts of preduals of von Neumann algebras (see Section 3.3). For a Banach lattice with an order continuous norm, Theorem 2.1.14 is due to Derriennic and Krengel [24].

2.1.10 The following two lemmas play a central role in the investigation of asymptotic properties of positive semigroups in uniformly order convex Banach spaces. These lemmas are taken from [32].

Lemma 2.1.15. *Let X be a uniformly order convex Banach space, $J = \mathbb{N}$ or $J = \mathbb{R}_+$, $(x_t)_{t\in J}$ be a family in X_+, and let $P_t : X_+ \to X_+$, $t \in J$, be non-expansive mappings (this means that $\|P_t x\| \le \|x\|$ for all $x \in X_+$ and $t \in J$). Let $(t_n)_{n=1}^\infty$ be a sequence in J converging to ∞, and assume that*

(i) $\lim\limits_{n\to\infty} P_{t_n-t_m}x_{t_m}$ *exists for all $m \in \mathbb{N}$, and*

(ii) $\lim\limits_{n\to\infty} \mathrm{dist}(P_{t_n-t_m}x_{t_m} - x_{t_n}, X_+) = 0$ *for all $m \in \mathbb{N}$.*

Then the sequence $(x_{t_n})_{n=1}^\infty$ is norm convergent.

Proof. By (i), if $m \in \mathbb{N}$ is fixed then the sequence $(P_{t_n-t_m}x_{t_m})_{n=1}^\infty$ is bounded. Assumption (ii) implies that there exist $z_n \in E$, $n \in \mathbb{N}$, such that

$$0 \le x_{t_n} \le P_{t_n-t_m}x_{t_m} + z_n \quad \text{and} \quad \lim_{n\to\infty}\|z_n\| = 0. \tag{2.9}$$

Then $(x_{t_n})_{n=1}^\infty$ is also bounded. Let $\alpha := \lim\inf\limits_{n\to\infty}\|x_{t_n}\| < \infty$. For fixed $\varepsilon > 0$, choose $m \in \mathbb{N}$ such that $\|x_{t_m}\| < \alpha+\varepsilon$. Since $P_{t_n-t_m}$ is non-expansive, (2.9) yields

$$\begin{aligned}\limsup_{n\to\infty}\|x_{t_n}\| &\le \limsup_{n\to\infty}\|P_{t_n-t_m}x_{t_m}\| \\ &\le \alpha+\varepsilon.\end{aligned}$$

Thus $\lim\sup\limits_{n\to\infty}\|x_{t_n}\| \le \alpha$, and hence $\alpha = \lim\limits_{n\to\infty}\|x_{t_n}\|$ exists.

Let $M := \sup_n \|x_{t_n}\| < \infty$ and fix $\varepsilon > 0$. Choose $\delta > 0$ corresponding to M and ε as in Lemma 2.1.7. There exists $r \in \mathbb{N}$ such that

$$\big|\|x_{t_n}\| - \|x_{t_m}\|\big| < \delta \quad \text{for} \quad n, m \ge r.$$

Assumption (i) implies that there is $r_1 > r$ such that

$$\|P_{t_n-t_r}x_{t_r} - P_{t_m-t_r}x_{t_r}\| < \varepsilon \quad \text{for} \quad n, m \ge r_1.$$

By assumption (ii), there exists $r_2 \ge r_1$ such that

$$\mathrm{dist}(P_{t_n-t_r}x_{t_r} - x_{t_n}, X_+) < \delta \quad \text{for} \quad n \ge r_2.$$

In particular, we find $w_n \in X, n \ge r_2$, such that

$$0 \le x_{t_n} \le P_{t_n-t_r}x_{t_r} + w_n \quad \text{and} \quad \|w_n\| < \delta.$$

Then

$$\begin{aligned}\|x_{t_n}\| - \delta &< \|P_{t_n - t_r} x_{t_r}\| \le \|x_{t_r}\| \\ &< \|x_{t_n}\| + \delta \qquad (n \ge r_2)\,,\end{aligned}$$

and hence

$$\big| \|P_{t_n - t_r} x_{t_r}\| - \|x_{t_n}\| \big| < \delta \quad \text{for} \quad n \ge r_2.$$

From Lemma 2.1.7, we get

$$\|P_{t_n - t_r} x_{t_r} - x_{t_n}\| < \varepsilon \quad \text{for} \quad n \ge r_2.$$

Thus, if $n, m \ge r_2$, then

$$\begin{aligned}\|x_{t_n} - x_{t_m}\| &\le \|x_{t_n} - P_{t_n - t_r} x_{t_r}\| + \|P_{t_n - t_r} x_{t_r} - P_{t_m - t_r} x_{t_r}\| \\ &\quad + \|P_{t_m - t_r} x_{t_r} - x_{t_m}\| \\ &< 3\varepsilon.\end{aligned}$$

Hence $(x_{t_n})_{n=1}^{\infty}$ is a Cauchy sequence, and the assertion follows. □

Lemma 2.1.16. *Let X be a uniformly order convex Banach space, let $J = \mathbb{N}$ or $J = \mathbb{R}_+$, $(x_t)_{t\in J}$ be a family in X_+, and let $P_t : X_+ \to X_+$, $t \in J$, be non-expansive mappings. Let $(t_n)_{n=1}^{\infty}$ be a sequence in J converging to ∞, and assume that*

(i) *the set $\{P_{t_n - t_m} x_{t_m} : \ n \in \mathbb{N}\}$ is relatively compact for all natural m,*

and

(ii) $\lim_{n\to\infty} \operatorname{dist}(P_{t_n - t_m} x_{t_m} - x_{t_n}, X_+) = 0$ *for all natural m.*

Then there is a convergent subsequence $(x_{s_n})_{n=1}^{\infty}$ of $(x_t)_{t\in J}$.

Proof. By the diagonal sequence argument, we can choose a subsequence $(s_n)_{n=1}^{\infty}$ of $(t_n)_{n=1}^{\infty}$ such that the sequence $(P_{s_n - t_m} x_{t_m})_{n=1}^{\infty}$ is convergent for each $m \in \mathbb{N}$. Thus $(P_{s_n - s_m} x_{s_m})_{n=1}^{\infty}$ converges for each $m \in \mathbb{N}$. Moreover,

$$\lim_{n\to\infty} \operatorname{dist}(P_{s_n - s_m} x_{s_m} - x_{s_n}, X_+) = 0 \qquad (\forall m \in \mathbb{N}).$$

Lemma 2.1.15 with $(t_n)_{n=1}^{\infty}$ replaced by $(s_n)_{n=1}^{\infty}$ yields the assertion. □

2.1.11 Now we are in a position to prove the following result (cf. [32, Thm. 3.1]).

Theorem 2.1.17 (Emel'yanov–Kohler–Räbiger–Wolff). *Let X be a uniformly order convex Banach space. Assume that $\mathcal{S}$ is a one-parameter positive semigroup in X which is asymptotically dominated either*

a) *by a non-expansive mapping $P : X_+ \to X_+$ or*

b) *by a one-parameter strongly stable positive contractive semigroup* $\mathcal{T}$*.*

Then $\mathcal{S}$ *is strongly stable. Moreover* $\operatorname{rank}(P_{\mathcal{S}}) \leq \operatorname{rank}(P_{\mathcal{T}})$.

Proof. a) Let $x \in X_+$ and let $(t_n)_{n=1}^{\infty}$ be a sequence in J such that $\lim_{n\to\infty} t_n = \infty$. For $t \in J$, set $x_t := S_t x$ and $P_t := P$. Then $(x_t)_{t\in J}$, $(P_t)_{t\in J}$, and $(t_n)_{n=1}^{\infty}$ satisfy the assumptions of Lemma 2.1.15, and hence $(x_{t_n})_{n=1}^{\infty}$ is convergent. Since this is true for every sequence $(t_n)_{n=1}^{\infty}$ in J converging to ∞, the stability of $\mathcal{S}$ follows.

b) Let $Px := \lim_{t\to\infty} T_t x,\ x \in X_+$. Then $P : X_+ \to X_+$ is non-expansive, and $\mathcal{S}$ is asymptotically dominated by P. Now the assertion follows from a). □

2.1.12 We finish with the following result on the inheritance of almost periodicity under asymptotic domination (cf. [32, Thm.3.3]).

Theorem 2.1.18. *Let* X *be a uniformly order convex Banach space. Assume that* $\mathcal{S}$ *is a one-parameter positive semigroup in* X *which is asymptotically dominated by a one-parameter almost periodic positive contractive semigroup* $\mathcal{T}$*. Then* $\mathcal{S}$ *is almost periodic.*

Proof. Let $x \in X_+$ and let $(t_n)_{n=1}^{\infty}$ be a sequence in J such that $\lim_{n\to\infty} t_n = \infty$. Set $x_t := S_t x$ and $P_t := T_t,\ t \in J$. Our assumptions imply that $(x_t)_{t\in J}$, $(P_t)_{t\in J}$, and $(t_n)_{n=1}^{\infty}$ fulfill the conditions of Lemma 2.1.16. Thus $(S_{t_n}x)_{n=1}^{\infty}$ has a convergent subsequence, and hence $\mathcal{S}$ is almost periodic. □

Related Results and Notes

2.1.13 It would be interesting to construct an ordered Banach space with a normal positive cone which is not strongly normal.

In Proposition 2.1.4, we can say even more about Banach lattices, namely that $C = 1$.

Exercise 2.1.19. Let E be a Banach lattice. Show that

$$\operatorname{dist}([0,x]\,,\,[0,y]) \leq \|x-y\| \quad (\forall x,y \in E_+). \tag{2.10}$$

It seems to be that the condition (2.10) characterizes Banach lattices among ordered Banach spaces.

Question 2.1.20. Let E be an ordered Banach space that satisfies the condition (2.10). Is E a Banach lattice?

2.1.14 A special case of Theorem 2.1.8 was proved in [101] for a one-parameter discrete semigroup of positive contractions in a Banach lattice. Then in [50], it was extended for any one-parameter discrete semigroup of positive operators in a Banach lattice. The main idea in [37] (and in [40]), where this theorem was obtained in its general form, is to find an extension of those results on the non-commutative L^1-spaces, which are far away from Banach lattices (see Section 3.3 for details). Note that without the assumption that the closure of the convex hull of the orbit $\mathcal{T}y := \{T_t y\}_{t\in J}$ contains a $\mathcal{T}$-invariant point, the statement of Theorem 2.1.8 is not true. To show this, it is enough to take the operator T_α in the Banach lattice c_0 from Example 1.3.8 with $\alpha > 0$.

The following result is an easy consequence of Theorem 2.1.8.

Theorem 2.1.21. *Let $\mathcal{T} = (T_t)_{t\in J}$ be an abelian positive operator semigroup in a strongly normal Banach space X with a strong order unit $\mathbb{1}$. If the limit*

$$w = \lim_{\tau\to\infty} \mathcal{A}_\tau^{\mathcal{T}} \mathbb{1}$$

exists, then

$$[-Mw, Mw] \in \operatorname{Constr}(\mathcal{T}),$$

where $M = \sup_{t\in\mathcal{J}} \|T_t\|$. In particular, if $w = 0$, then $\lim_{t\to\infty} \|T_t\| = 0$.

Proof. By hypothesis, the order interval $[-M\mathbb{1}, M\mathbb{1}]$ is a constrictor of $\mathcal{T}$. Thus the first assertion follows from Theorem 2.1.8. If $w = 0$, then $\lim_{t\to\infty} \|T_t\mathbb{1}\| = 0$, which implies, in view of $T_t \geq 0$, that $\lim_{t\to\infty} \|T_t\| = 0$. □

In Section 3.3, we show (Theorem 3.3.16) that the self-adjoint part of any C^*-algebra has a strongly normal positive cone. Thus we can apply Theorem 2.1.21, and obtain the following corollary which seems to be interesting even for a commutative C^*-algebra. The second part is a generalization of the result of Groh and Neubrander [54, Satz 3.2], where the weak convergence of $\mathcal{T}\,\mathbb{1}$ to 0 is assumed in order to derive uniform convergence to 0. In fact, under the assumption that $T_t\mathbb{1}$ converges weakly to 0, by the Eberlein mean ergodic theorem, $w = 0$ in the corollary below. A result related to this second part was proved by Batty [16, Thm. 3] in a more general setting.

Corollary 2.1.22. *Let $\mathcal{T} = (T_t)_{t\geq 0}$ be a positive C_0-semigroup in a C^*-algebra $\mathcal{A}$ with the unit $\mathbb{1}$. If the limit*

$$w = \lim_{\tau\to\infty} \mathcal{A}_\tau^{\mathcal{T}} \mathbb{1}$$

exists, then $[-Mw, Mw]$ is a constrictor of the restriction of $\mathcal{T}$ onto the self-adjoint part $\mathcal{A}_{sa}$ of $\mathcal{A}$, where $M = \sup_{t\geq 0} \|T_t\|$. Moreover, if $w = 0$, then $\lim_{t\to\infty} \|T_t\| = 0$.

Proof. By the hypothesis, the interval $[-M\mathbb{1}, M\mathbb{1}]$ is a constrictor of the restriction $\mathcal{T}|_{\mathcal{A}_{sa}}$. Then the assertion follows from Theorem 2.1.21. If $w = 0$, then $\lim \|T_t\mathbb{1}\| = 0$ which implies, in view of positivity of $\mathcal{T}$, that $\lim_{t\to\infty} \|T_t\| = 0$. □

2.1.15 The following interesting question about the possibility of replacing $\mathcal{T}$ by $\mathcal{A}_t^{\mathcal{T}}$ in Theorem 2.1.8 is open even for the one-parameter case. According to Theorem 2.2.5, the answer is "yes" for a bounded one-parameter positive operator semigroup in a KB-space.

Open Problem 2.1.23. *Let X be a strongly normal Banach space, and let $\mathcal{T}$ be a one-parameter positive operator semigroup in X. We suppose that $\mathcal{T}$ satisfies*

$$\lim_{t\to\infty} \sup \operatorname{dist}(\mathcal{A}_t^{\mathcal{T}} x,\, [-y,y] + \eta B_X) = 0 \quad (\forall x \in B_X)$$

for some $y \in X_+$ and some real η, $0 \le \eta < 1$; and suppose that the closure of the convex hull of the orbit $\mathcal{T}y := \{T_t y\}_{t\in J}$ contains a $\mathcal{T}$-invariant point w. Does the interval $[-w,w]$ satisfy

$$\lim_{t\to\infty} \sup \operatorname{dist}(\mathcal{A}_t^{\mathcal{T}} x,\, [-w,w]) = 0 \quad (\forall x \in B_X)\,?$$

2.1.16 In the case when one-parameter semigroups $\mathcal{S}$ and $\mathcal{T}$ consist of linear operators acting in Banach lattices with order continuous norm (for example, in L^p-spaces, where $1 \le p < \infty$), which are uniformly order convex Banach spaces, Theorems 2.1.17, 2.1.18 will be generalized in Section 2.2, namely, they hold without the condition that $\mathcal{T}$ is non-expansive. The following example shows that the uniform order convexity condition is essential there.

Example 2.1.24. Let $E = c$ be the space of convergent sequences endowed with the sup-norm. Let $T = I_E$, and define $S \in \mathcal{L}_+(E)$ by

$$S(\xi) := \Big(\frac{k}{k+1}\xi_k\Big)_{k=1}^{\infty} \quad (\xi = (\xi_k)_{k=1}^{\infty} \in c).$$

Then $\mathcal{T} = (T^n)_{n=1}^{\infty}$ is strongly stable and contractive, and $\mathcal{S} = (S^n)_{n=1}^{\infty}$ is asymptotically dominated by $(T^n)_{n=1}^{\infty}$. However, $(S^n)_{n=1}^{\infty}$ is even not almost periodic.

2.1.17 There are several interesting results related to Theorem 2.1.21 about positive semigroups in $C(K)$. We mention only a few of them. An operator T in the ordered Banach space $X = C(K)$ is called *irreducible* if for each $f \in X_+$, $f \neq 0$, and each $a \in K$ there is $n \in \mathbb{N}$ with $T^n f(a) > 0$. By $\mathbb{1}_K$ we denote the function from $C(K)$ which is identically equal to 1. The following result is due to Jamison [58]; for the proof of it, we send the reader to [67, pp. 180–182].

Theorem 2.1.25 (Jamison). *Let K be a compact Hausdorff space, $f \in C(K)$, and let T be a positive irreducible operator in $C(K)$ such that $T(\mathbb{1}_K) = \mathbb{1}_K$. If T satisfies*

$$\lim_{n\to\infty} T^n(f)(a) = 0 \quad (\forall a \in K)$$

then $\lim\limits_{n\to\infty} \|T^n(f)\| = 0.$ □

Exercise 2.1.26. Let K be a compact Hausdorff space, and let T be a weakly almost periodic irreducible positive operator in $X = C(K)$ satisfying $T(\mathbb{1}_K) = \mathbb{1}_K$. Show that $\lim\limits_{n\to\infty} \|T^n f\| = 0$ for any $f \in X_r(T)$. Show that T is almost periodic.

Hint: See [67, p.182].

Exercise 2.1.27. Let K be a compact Hausdorff space, and let T be a positive operator in $X = C(K)$. Show that if $T^n(\mathbb{1}_K)$ converges to 0 pointwise, then $\lim\limits_{n\to\infty} \|T^n(\mathbb{1}_K)\| = 0$.

Hint: See the proof of [67, Thm.5.1.14].

2.2 Positive semigroups in Banach lattices

If $\mathcal{S}$ and $\mathcal{T}$ are one-parameter positive operator semigroups in a Banach lattice such that $\mathcal{S}$ is dominated by $\mathcal{T}$, then it is natural to ask which properties of $\mathcal{T}$ are inherited by $\mathcal{S}$. There are numerous results in this direction dealing with compactness, weak compactness, the Dunford–Pettis property, and others (see [4], [85], [131] for a survey of such results). In this section, we discuss asymptotic properties of dominated semigroups, such that the mean ergodicity, strong stability, and almost periodicity. Most of results can be formulated for positive operators as well as for positive C_0-semigroups in Banach lattices. We begin with the main definitions concerning Banach lattices and positive operators in them and then discuss the inheritance of mean ergodicity and (weak) almost periodicity under domination. Our notations are standard and follow to the books [5], [85], [110], and [131].

2.2.1 Let E be an (usually real) ordered Banach space. In this section, we everywhere assume that E is a *Banach lattice*, which means that $\sup(x, y)$ and $\inf(x, y)$ exist for any two elements $x, y \in E$, and that $\|x\| \leq \|y\|$ whenever $|x| \leq |y|$. We use notations $\vee$ and $\wedge$ for the lattice operations sup and inf and, as usual, we denote $x_+ := \sup(x, 0)$, $x_- := (-x)_+$, and $|x| := x_+ + x_-$ the positive part, negative part, and modulus of $x \in E$ respectively. Examples of Banach lattices are L^p-spaces $(1 \leq p \leq \infty)$, $C(K)$, c_0, etc.

A Banach lattice E is called *Dedekind complete* if any order bounded subset of E has a supremum. E is called σ-*Dedekind complete* if any countable order bounded subset of E has a supremum. Such Banach lattices as L^p-spaces $(1 \leq p \leq \infty)$ and c_0 are Dedekind complete. The Banach lattice c of all real-valued convergent sequences is even not σ-Dedekind complete. More generally, $C(K)$ is Dedekind complete if and only if every open set in the Hausdorff compact space K has open closure.

A subset A of the Banach lattice E is called an *ideal* if A is a vector subspace of E which satisfies:

$$x \in A,\ y \in E,\ |y| \leq |x| \Rightarrow y \in A.$$

An ideal B in E is called a *band* if for every increasing net in B that has a supremum u in E, the element u belongs to B. For example, $L^q[0,1]$ is an ideal in $L^p[0,1]$ whenever $q \geq p$, but if $q > p$ it is not a band.

A Banach lattice E has a *topological orthogonal system* (t.o.s. for short) if there exists a pairwise disjoint set A of positive elements of E such that the ideal generated by A is norm dense in E. If E has a t.o.s. consisting of one element, this element is called a *quasi-interior point* of E.

Another important condition on Banach lattices is the order continuity of the norm: the norm in E is called *order continuous* if any decreasing to zero net in E is norm convergent to zero. Such Banach lattices as L^p-spaces $(1 \leq p < \infty)$ and c_0 have order continuous norm. $C(K)$ has order continuous norm in the trivial case only, when K is finite. Any Banach lattice having an order continuous norm is Dedekind complete. A Banach lattice E has an order continuous norm if and only if E is an ideal in its bidual E^{**} (see [85]). It is easy to see that every uniformly order convex Banach lattice has order continuous norm (see [110, II.5.10, II.5.15]).

A set A in the Banach lattice E is called *almost order bounded* if for every $\varepsilon > 0$ there exists $x_\varepsilon \in E_+$ such that $\|(|x|-x_\varepsilon)_+\| \leq \varepsilon$ for all $x \in A$. In other words, for any $\varepsilon > 0$, there exists an interval $[-a_\varepsilon, a_\varepsilon]$ such that $A \subseteq [-a_\varepsilon, a_\varepsilon] + \varepsilon B_E$ (cf. Definition 2.1.1).

A Banach lattice E is called a *KB-space* if any norm bounded increasing net has a supremum and converges to it in the norm. Examples of KB-spaces are reflexive Banach lattices and L^1-spaces. c_0 is not a KB-space. Any KB-space has an order continuous norm and it can be shown that E is a KB-space if and only if E is a band in its bidual E^{**} (see [85]).

Any positive operator in a Banach lattice E is continuous (the proof of this simple property can be found in any book about Banach lattices). By $\mathcal{L}_+(E)$ and E^*_+ we denote the set of positive operators and the set of positive linear forms in E, respectively. An operator T is called *regular* if it can be written as a difference of two positive operators.

A linear operator T from a Banach lattice E to a Banach lattice F is called a *lattice homomorphism* (*Riesz homomorphism*) if $T(|x|) = |T(x)|$ for any $x \in E$. Any lattice homomorphism $T : E \to F$ is positive, and if T^{-1} exists, then $T^{-1} : F \to E$ is a lattice homomorphism as well; in this case T is called a *lattice isomorphism* (*Riesz isomorphism*).

If T is a positive operator in E, then $x \in E$ is called a *positive fixed vector of maximal support* if $x \in \mathrm{Fix}(T) \cap E_+$ and every $y \in \mathrm{Fix}(T) \cap E_+$ is contained in the band generated by x.

A linear functional $x' \in E^*_+$ is said to be *strictly positive* if $\langle x', x\rangle > 0$ for all $x \in E_+ \setminus \{0\}$.

2.2.2 Now we investigate asymptotic properties of a one-parameter asymptotically dominated semigroup in a Banach lattice. We shall see that (compare with Section 2.1) the uniform order convexity of an ordered Banach space can be replaced by a considerably weaker condition of the order continuity of the norm of a Banach lattice.

Note that we always have in a Banach lattice E,

$$\operatorname{dist}(x, E_+) = \|(-x)_+\| \qquad (\forall x \in E),$$

and hence a positive semigroup $\mathcal{S} = (S_t)_{t\in J}$ in E is asymptotically dominated by a positive semigroup $\mathcal{T} = (T_t)_{t\in J}$ in E if and only if

$$\lim_{t\to\infty} \|(S_t x - T_t x)_+\| = 0 \quad \text{for all} \quad x \in E_+.$$

We need the following construction, which can be found in [110, II.8, Ex.1].

Let $x' \in E_+^*$ be a positive linear functional in the Banach lattice E. Then

$$N(x') := \{x \in E : \ \langle x', |x|\rangle = 0\}$$

is a closed ideal in E, and

$$\|x + N(x')\| := \langle x', |x|\rangle, \ x \in E,$$

defines a lattice norm on the quotient space $E/N(x')$. The completion (E, x') of $E/N(x')$ with respect to this norm is a Banach lattice. The norm is additive on $(E, x')_+$, and hence (E, x') is uniformly order convex. The quotient map

$$q : E \to E/N(x')$$

defines the lattice homomorphism $j_{x'} : E \to (E, x')$. If $T \in \mathcal{L}_+(E)$ is an operator such that $T'x' \leq x'$, then the ideal $N(x')$ is T-invariant, and T induces the positive contraction $\tilde{T}$ in (E, x') given by

$$\tilde{T} j_{x'} x = j_{x'} T x, \quad (x \in E)\,.$$

The following lemma connects convergence of an almost order bounded sequence in E with convergence in (E, x') (see for the proof [85, 2.4.8] or [100, Lm.3.8]).

Lemma 2.2.1. *Let $(x_n)_{n=1}^\infty$ be an almost order bounded sequence in a Banach lattice E with order continuous norm, and let x' be a strictly positive linear functional in E. Then $(x_n)_{n=1}^\infty$ is convergent in E if and only if $(j_{x'} x_n)_{n=1}^\infty$ is convergent in (E, x').* □

2.2.3 We come to one of the main results of this section about the inheritance of strong stability and almost periodicity of an asymptotically dominated semigroup in a Banach lattice (see [32]).

Theorem 2.2.2 (Emel'yanov–Kohler–Räbiger–Wolff). *Let E be a Banach lattice with order continuous norm and let $\mathcal{S}$ and $\mathcal{T}$ be one-parameter positive operator semigroups in E such that $\mathcal{S}$ is asymptotically dominated by $\mathcal{T}$. If $\mathcal{T}$ is almost periodic then $\mathcal{S}$ is almost periodic. Moreover, if the Jacobs–Deleeuw–Glicksberg projection $P_{\mathcal{T}}$ has finite rank, then the corresponding projection $P_{\mathcal{S}}$ has finite rank and* $\operatorname{rank}(P_{\mathcal{S}}) \le \operatorname{rank}(P_{\mathcal{T}})$.

Proof. First we consider the case $J = \mathbb{N}$. By passing to an equivalent norm, we may assume that $\mathcal{S}$ consists of contractions. Now fix $x \in E_+$. Let F be the closed ideal in E generated by

$$M = \{S_{m_k} \circ T_{n_k} \circ \cdots \circ S_{m_0} \circ T_{n_0}\, x : \; m_0, n_0, \ldots, m_k, n_k \in \mathbb{N}, \; k \in \mathbb{N}\}.$$

Then F is invariant for $\mathcal{S}$ and $\mathcal{T}$, and $u = \sum_{n=1}^{\infty} 2^{-n} x_n$ is a weak order unit of F, where $\{x_n : \; n \in \mathbb{N}\}$ is an enumeration of M. Thus, by restricting $\mathcal{S}$ and $\mathcal{T}$ to F, we may assume that E contains a weak order unit. By [79, 1.b.15], there is a strictly positive linear functional $z' \in E^*_+$. Since $\mathcal{T}$ is almost periodic, $\mathcal{T}$ is mean ergodic with the mean ergodic projection $P_{\mathcal{T}}$. Consider the closed ideal

$$I := \{z \in E : \; P_{\mathcal{T}}|z| = 0\}\,.$$

The order continuity of the norm implies that I is a projection band (see [85, 2.4.4]); hence $E = I \oplus I^{\perp}$, where

$$I^{\perp} = \{z \in E : \; |z| \wedge |y| = 0$$

for all $y \in I\}$. Let R be the band projection from E onto $I^{\perp}$. Note that $0 \le R \le I_E$ and R is a lattice homomorphism (see [110, II.2.9]). We claim that $\{S_n z : \; n \in \mathbb{N}\}$ is relatively compact in E for every $z \in E_+$. It suffices to show that this is true for $z \in I_+$ and $z \in (I^{\perp})_+$, respectively.

1-st case: $z \in I_+$. Since $\mathcal{T}$ commutes with $P_{\mathcal{T}}$, the ideal I is $\mathcal{T}$-invariant. Denote by $\mathcal{T}_|$ the restriction of $\mathcal{T}$ to I. Let $w \in I$ and $\alpha : \mathbb{N} \to \Gamma$ be a character such that $T_n w = \alpha(n) w, \; n \in \mathbb{N}$. Then $|w| \le T_n |w|, \; n \in \mathbb{N}$, and hence $|w| \le P_{\mathcal{T}}|w| = 0$ which implies $w = 0$. Since $\mathcal{T}_|$ is almost periodic, it follows from Theorem 1.1.4 that $\mathcal{T}_|$ converges strongly to 0. In particular, $\lim\limits_{n\to\infty} T_n\, z = 0$. Since $\mathcal{S}$ is asymptotically dominated by $\mathcal{T}$, we obtain $\lim\limits_{n\to\infty} S_n\, z = 0$.

2-nd case: $z \in (I^{\perp})_+$. By assumption, $\{T_n z : \; n \in \mathbb{N}\}$ is relatively compact and hence almost order bounded. Since $\mathcal{S}$ is asymptotically dominated by $\mathcal{T}$, the set $\{S_n z : \; n \in \mathbb{N}\}$ is almost order bounded as well. Let $x' := P'_{\mathcal{T}} z'$. Then $T'_n x' = x', \; n \in \mathbb{N}$. The induced representation $\tilde{\mathcal{T}} = (\tilde{T}_n)_{n=1}^{\infty}$ in (E, x') determined by $\tilde{T}_n j_{x'} = j_{x'} T_n, \; n \in \mathbb{N}$, is positive, contractive, and almost periodic. Moreover,

$$\begin{aligned}
&\lim_{n\to\infty} \operatorname{dist}(\tilde{T}_{n-m}(j_{x'} S_m z) - j_{x'} S_n z, (E, x')_+) \\
&= \lim_{n\to\infty} \|[j_{x'} S_n z - j_{x'} T_{n-m} \circ S_m\, z]_+\| \\
&= \lim_{n\to\infty} \langle x', (S_n z - T_{n-m} \circ S_m\, z)_+ \rangle \\
&= 0.
\end{aligned}$$

Apply Theorem 2.1.18 to the sequence $(j_{x'}S_n z)_{n=1}^{\infty} \subseteq (E, x')_+$, operators $\tilde{T}_n \in \mathcal{L}_+((E, x'))$, and every sequence $(t_n)_{n=1}^{\infty}$ in $\mathbb{N}$ converging to ∞. We show that the set $\{j_{x'}S_n z : \ n \in \mathbb{N}\}$ is relatively compact in (E, x'). From

$$\begin{aligned} \|j_{x'}R \circ S_n\, z - j_{x'}R \circ S_m\, z\|_{(E,x')} &= \langle x', |R \circ S_n\, z - R \circ S_m\, z|\rangle \\ &\leq \langle x', |S_n z - S_m z|\rangle \\ &= \|j_{x'}S_n z - j_{x'}S_m z\|_{(E,x')}, \quad (n, m \in \mathbb{N}), \end{aligned}$$

we derive that

$$\{j_{x'}R \circ S_n\, z : \ n \in \mathbb{N}\}$$

is relatively compact in (E, x'). Since $x'|_{I^\perp}$ is strictly positive and

$$\{R \circ S_n\, z : \ n \in \mathbb{N}\}$$

is almost order bounded, Lemma 2.2.1 implies that $\{R \circ S_n\, z : \ n \in \mathbb{N}\}$ is relatively compact in $I^\perp \subseteq E$.

Now we show that $\{S_n z : \ n \in \mathbb{N}\}$ is relatively compact. Fix $\varepsilon > 0$. Since $\{S_n z : \ n \in \mathbb{N}\}$ is almost order bounded, there exists $y \in E_+$ such that

$$\|(|S_k z - S_l z| - y)_+\| \leq \varepsilon \quad (\forall k, l \in \mathbb{N}).$$

By the contractivity of the operators S_n and $I_E - R$, we obtain

$$\begin{aligned} \|S_n(S_k z - S_l z)\| - \|R(S_k z - S_l z)\| &\leq \|S_n(S_k z - S_l z)\| - \|S_n \circ R(S_k z - S_l z)\| \\ &\leq \|S_n \circ (I_E - R)(S_k z - S_l z)|\| \\ &\leq \|S_n \circ (I_E - R)\, y\| + \varepsilon. \end{aligned}$$

From the first case, we know that $\lim_{n\to\infty} S_n \circ (I_E - R)\, y = 0$. Hence there exists $n_0 \in \mathbb{N}$ such that

$$\|S_{n_0}(S_k z - S_l z)\| \leq \|R(S_k z - S_l z)\| + 2\varepsilon \quad (\forall k, l \in \mathbb{N}). \tag{2.11}$$

Since $\{R \circ S_n\, z : \ n \in \mathbb{N}\}$ is relatively compact, there exists $m_0 \in \mathbb{N}$ such that

$$\{R \circ S_n\, z : \ n \in \mathbb{N}\} \subseteq \{R \circ S_m\, z : \ 0 \leq m \leq m_0\} + \varepsilon B_E.$$

Now let $n > m_0 + n_0$. There exists $0 \leq m \leq m_0$ such that

$$\|R \circ S_{n-n_0}\, z - R \circ S_m\, z\| \leq \varepsilon.$$

By (2.11), we have

$$\begin{aligned} \|S_n z - S_{m+n_0} z\| &\leq \|R \circ S_{n-n_0}\, z - R \circ S_m\, z\| + 2\varepsilon \\ &\leq 3\varepsilon. \end{aligned}$$

Thus
$$\{S_n z:\ n\in\mathbb{N}\}\subseteq\{S_m z:\ 0\le m\le m_0+n_0\}+3\varepsilon B_E,$$
i.e., $\{S_n z:\ n\in\mathbb{N}\}$ is totally bounded and hence relatively compact. This proves the assertion for $J=\mathbb{N}$.

If $J=\mathbb{R}_+$, it follows from above that $\{S_n x:\ n\in\mathbb{N}\}$ is relatively compact for each $x\in E$. The strong continuity of $\mathcal{S}$ and Proposition 1.1.3 imply that $\{S_t x:\ t\ge 0\}$ is relatively compact for each $x\in E$. The rest of the proof follows from Proposition 2.1.3. □

2.2.4 From Theorem 2.2.2, we can deduce the following result (see [32]) on the inheritance of strong stability.

Theorem 2.2.3 (Emel'yanov–Kohler–Räbiger–Wolff). *Let E be a Banach lattice with order continuous norm and let $\mathcal{S}$ and $\mathcal{T}$ be one-parameter positive operator semigroups in E such that $\mathcal{S}$ is asymptotically dominated by $\mathcal{T}$. If $\mathcal{T}$ is strongly stable, then $\mathcal{S}$ is strongly stable, and* $\operatorname{rank}(P_{\mathcal{S}})\le\operatorname{rank}(P_{\mathcal{T}})$.

Proof. By Theorem 2.2.2, the semigroup $\mathcal{S}$ is almost periodic. The corresponding Jacobs–Deleeuw–Glicksberg projections $P_{\mathcal{S}}$ and $P_{\mathcal{T}}$ satisfy
$$0\le S_t\circ P_{\mathcal{S}}\le T_t\circ P_{\mathcal{T}}=P_{\mathcal{T}}\qquad(\forall t\in J).$$
First of all, remark that
$$P_{\mathcal{S}}\circ P_{\mathcal{T}}\circ P_{\mathcal{S}}\,x\ge P_{\mathcal{S}}^3\,x=P_{\mathcal{S}}\,x\,,$$
and
$$\begin{aligned}P_{\mathcal{S}}\circ P_{\mathcal{T}}\circ P_{\mathcal{S}}\,x &= P_{\mathcal{S}}\circ T_t\circ P_{\mathcal{T}}\circ P_{\mathcal{S}}\,x\\ &\ge P_{\mathcal{S}}\circ T_t\circ P_{\mathcal{S}}\,x\\ &\ge P_{\mathcal{S}}\circ S_t\circ P_{\mathcal{S}}\,x\\ &= S_t\circ P_{\mathcal{S}}\,x\end{aligned}$$
for all $x\in E_+$ and $t\in J$. Thus,
$$P_{\mathcal{S}}\circ P_{\mathcal{T}}y\ge\ y,\quad P_{\mathcal{S}}\circ P_{\mathcal{T}}y\ge S_t\,y\quad(\forall\ 0\le x\in P_{\mathcal{S}}(E))$$
for all $t\in J$. So, we have
$$C=P_{\mathcal{S}}\circ P_{\mathcal{T}}|_{P_{\mathcal{S}}(E)}\ge I|_{P_{\mathcal{S}}(E)},\ S_t|_{P_{\mathcal{S}}(E)}\quad(\forall t\in J),$$
with respect to the order in $\mathcal{L}(P_{\mathcal{S}}(E))$. Then
$$C^{**}\ge I\vee S_t^{**}\in\mathcal{L}_+(P_{\mathcal{S}}(E)^{**}).$$
The supremum $I\vee S_t^{**}$ exists in $\mathcal{L}(P_{\mathcal{S}}(E)^{**})$ since the Banach lattice $P_{\mathcal{S}}(E)^{**}$ is Dedekind complete.

Let $S_t^{**} \not\leq I$. Then $I \vee S_t^{**} = I + A$, where

$$0 < A \in \mathcal{L}(P_{\mathcal{S}}(E)^{**}).$$

Consequently,

$$C^{**n} \geq (I+A)^n \geq I + nA \qquad (\forall n \in \mathbb{N}).$$

Take an operator $C \circ P_{\mathcal{S}} \in \mathcal{L}_+(E)$. Then

$$\begin{aligned} 0 \leq C^n \circ P_{\mathcal{S}} &= (C \circ P_{\mathcal{S}})^n \\ &\leq P_{\mathcal{T}}^{3n} \\ &= P_{\mathcal{T}} \qquad (\forall n \in \mathbb{N}). \end{aligned}$$

Thus the orbit $\{C^n y\}_{n=1}^{\infty}$ is bounded in E and, therefore, in $P_{\mathcal{S}}(E)$ for any $y \in P_{\mathcal{S}}(E)$. Then the set $\{C^n\}_{n=1}^{\infty} \subseteq \mathcal{L}(P_{\mathcal{S}}(E))$ is bounded in the operator norm by some $M > 0$. Then

$$\begin{aligned} n\|Ax\| &\leq \|x + nAx\| \\ &= \|(I + nA)x\| \\ &\leq \|C^{**n}x\| \\ &\leq \|C^{**n}\| \cdot \|x\| \\ &\leq M\|x\| \end{aligned}$$

for all $0 \leq x \in P_{\mathcal{S}}(E)^{**}$ and for all $n \in \mathbb{N}$, which is impossible, since $A \neq 0$. The obtained contradiction shows that $S_t^{**} \leq I$ on $P_{\mathcal{S}}(E)^{**}$.

Thus $\mathcal{S}|_{P_{\mathcal{S}}(E)} \leq I|_{P_{\mathcal{S}}(E)}$ and $\mathcal{L}(P_{\mathcal{S}}(E))$. This inequality and the fact that the spectrum $\sigma(S_t|_{P_{\mathcal{S}}(E)})$ lies on the unit circle imply that $S_t = Id$ on $P_{\mathcal{S}}(E)$ for all $t \in J$. Thus the semigroup $\mathcal{S}$ is strongly stable. □

2.2.5 It was proved in [39] that if T is a Markov operator in L^1-space, then T is mean ergodic and satisfies $\dim \operatorname{Fix}(T) < \infty$, whenever there exist a function $h \in L^1_+$ and real η, $0 \leq \eta < 1$, such that

$$\lim_{n\to\infty} \left\| \left(h - \frac{1}{n} \sum_{k=0}^{n-1} T^k f \right)_+ \right\| \leq \eta$$

for every density f. We turn back to this theorem in Section 3.1. This result was extended to any positive power bounded operator in a KB-space in [8]. Moreover, it was shown that this property of positive operators characterizes KB-spaces among σ-Dedekind complete Banach lattices.

Theorem 2.2.4 (Alpay–Binhadjah–Emel'yanov–Ercan). *Let E be a KB-space, let T be a positive power bounded operator in E, let W be a weakly compact subset of E, and let $\eta \in \mathbb{R}$, $0 \leq \eta < 1$, be such that*

$$\lim_{n\to\infty} \operatorname{dist}(\mathcal{A}_n^T x, W + \eta B_E) = 0$$

for any $x \in B_E$. Then T is mean ergodic.

Proof. Without loss of generality we may assume that E has a quasi-interior point. Indeed, for any $x \in E$, $x \neq 0$, we consider the closed order ideal F generated by $\{T^n|x| : n \geq 0\}$ instead of E. Then F is a KB-space [110, Prop.II.5.15] with a quasi-interior point $\sum_{n \geq 0} 2^{-n} T^n |x|$ and $T(F) \subseteq F$. Moreover, F is a projection band in E [85, Cor.2.2.4]. If $P : E \longrightarrow F$ denotes the corresponding band projection, then

$$\lim_{n\to\infty} \operatorname{dist}(\mathcal{A}_n^T z, P(W) + \eta B_F) = 0 \quad (\forall z \in B_F).$$

Since $\|P\| = 1$ and $P(W)$ is weakly compact in F, the restriction $T|_F$ satisfies the assumptions of the theorem. Thus, to show that $(\mathcal{A}_n^T x)_{n=1}^{\infty}$ converges, it is enough to show that $T|_F$ is mean ergodic. Hence we may assume that E has a quasi-interior point, say e.

There are two alternative cases:

1-st case: $(\mathcal{A}_n^{T'} x')_{n=1}^{\infty}$ is a $\sigma(E', E)$-null sequence for each $x' \in E'$. Then $(\mathcal{A}_n^T x)_{n=1}^{\infty}$ converges weakly to 0 for each $x \in E$ and hence, by Theorem 1.1.7, $(\mathcal{A}_n^T)_{n=1}^{\infty}$ converges strongly to 0. Hence T is mean ergodic.

2-nd case: There is $x' \in E'_+$ such that $(\mathcal{A}_n^{T'} x')_{n=1}^{\infty}$ is not $\sigma(E', E)$-convergent to 0. Let $0 \neq y' \in E'_+$ be a $\sigma(E', E)$-cluster point of $(\mathcal{A}_n^{T'} x')_{n=1}^{\infty}$. We may assume $\|y'\| = 1$. Then, for all $\varepsilon > 0$, there exists n with

$$\langle y', x\rangle - \langle \mathcal{A}_n^{T'} x', x\rangle < \varepsilon \ \ \& \ \ \langle T'y', x\rangle - \langle T'\mathcal{A}_n^{T'} x', x\rangle < \varepsilon.$$

Combining these estimates, we arrive at

$$\langle y', x\rangle - \langle T'y', x\rangle < 2\varepsilon\,,$$

but ε and x were chosen arbitrarily, so $T'y' = y'$.

Fix $\varepsilon > 0$ satisfying $\eta + \varepsilon < 1$, choose $x \in B_E \cap E_+$ such that $\langle y', x\rangle > 1 - \varepsilon$. Let $x'' \in E''_+$ be a $\sigma(E'', E')$-cluster point of $(\mathcal{A}_n^T x)_{n=1}^{\infty}$. Then, by the same arguments as before, $T''x'' = x''$.

Since W is weakly compact in E and $\lim\limits_{n\to\infty} \operatorname{dist}(\mathcal{A}_n^T x, W + \eta B_E) = 0$, we obtain $x'' \in W + \eta B_{E''}$. Moreover,

$$\langle y', x''\rangle = \langle y', x\rangle > 1 - \varepsilon.$$

(Since x'' is a $\sigma(E'', E')$-cluster point of $(\mathcal{A}_n^T x)_{n=1}^{\infty}$ then for every $\delta > 0$ there exists n_δ such that

$$\langle y', x''\rangle - \langle y', \mathcal{A}_{n_\delta}^T x\rangle < \delta\,.$$

Thus we have $\langle y', x''\rangle - \langle \mathcal{A}_{n_\delta}^{T'} y', x\rangle < \delta$, and since $T'y' = y'$,

$$\langle y', x''\rangle - \langle y', x\rangle < \delta\,.$$

By arbitrariness of δ, $\langle y', x''\rangle = \langle y', x\rangle$.)

Let P be the band projection from E'' onto E (such a projection exists because E is a KB-space). Then $(I_{E''} - P)x'' \in \eta B_{E''}$, and hence

$$\langle y', Px'' \rangle = \langle y', x'' \rangle - \langle y', (I_{E''} - P)x'' \rangle > 1 - \varepsilon - \eta > 0.$$

From

$$\begin{aligned} Px'' + (I_{E''} - P)x'' = x'' &= T''x'' \\ &= x'' \\ &= T \circ P\,x'' + T'' \circ (I_{E''} - P)\,x'' \in E_+ + E''_+, \end{aligned}$$

and from the fact that Px'' is the biggest part of x'' in E_+, we get

$$0 \le T \circ P\,x'' \le Px''\,.$$

Hence $(T^n \circ P\,x'')_{n=1}^{\infty}$ is a decreasing sequence in E_+. Since E has order continuous norm, $z := \lim_{n\to\infty} T^n \circ P\,x'' \in E_+$ exists. Clearly $Tz = z$ and, from

$$\langle y', z \rangle = \langle y', Px'' \rangle > 0\,,$$

it follows that $z \neq 0$. Hence $\mathrm{Fix}(T) \cap E_+ \neq \{0\}$.

Now the existence of a quasi-interior point e in E implies the existence of a strictly positive linear functional ψ in E [79, Thm.1.b.15]. For $x \in E$, let P_x be the band projection from E onto the band generated by x. Set

$$\alpha := \sup_{x \in \mathrm{Fix}(T) \cap E_+} \langle \psi, P_x e \rangle > 0.$$

Choose $x_n \in \mathrm{Fix}(T) \cap E_+$, $n \in N$, $\|x_n\| \le 1$, with $\alpha = \lim_n \langle \psi, P_{x_n} e \rangle$. Let $u := \sum_n 2^{-n} x_n$. Then $u \in \mathrm{Fix}(T) \cap E_+$, $P_u \ge P_{x_n}$ for all $n \in N$, and hence $\langle \psi, P_u e \rangle = \alpha$. Let now $x \in \mathrm{Fix}(T) \cap E_+$. Clearly $P_{u+x} \ge P_x$ and $P_{u+x} \ge P_u$. From

$$\alpha = \langle \psi, P_u e \rangle \le \langle \psi, P_{u+x} e \rangle \le \alpha$$

and the strict positivity of ψ, we obtain $P_u e = P_{x+u} e$. Since e is a quasi-interior point, $P_u = P_{x+u}$. From $P_{u+x} \ge P_x$, it follows that $P_u \ge P_x$. Thus $u \in \mathrm{Fix}(T) \cap E_+$ has a maximal support.

Denote by B_u the projection band generated by u.

$$B_u = \overline{\bigcup_{n=1}^{\infty} [-nu, nu]}$$

by the order continuity of the norm in E. Denote $Q = I_E - P_u$ and $S = Q \circ T$. Since $T \circ P_u = P_u \circ T \circ P_u$, an easy calculation shows that

$$Q \circ T = Q \circ T \circ Q$$

(and then $(Q \circ T \circ Q)^n = (Q \circ T)^n = Q \circ T^n$ for all n).

Show that the sequence $(Q \circ \mathcal{A}_n^T)_{n=1}^{\infty}$ is strongly convergent to 0. If not then $\mathcal{A}_n^S \not\to 0$, and, as in the **2-nd case**, there exists $y' \in \mathrm{Fix}(S') \cap E'_+$, $y' \neq 0$. From

$$\begin{aligned} y' = T' \circ Q'\, y' &= Q' \circ T' \circ Q'\, y' \\ &= Q' \circ S'\, y' \\ &= Q'\, y'\,, \end{aligned}$$

we obtain that $y' \in \mathrm{Fix}(T') \cap E'_+$. Again, as in the **2-nd case**, for this y', there exists $y \in \mathrm{Fix}(T) \cap E_+$ such that $\langle y', y \rangle > 0$. Then

$$\langle y', Qy \rangle = \langle Q'y', y \rangle = \langle y', y \rangle > 0\,.$$

Hence $(Id_E - P_u)y = Qy \neq 0$, i.e., $y \notin B_u$. This contradicts the fact that u has a maximal support. Thus $Q \circ \mathcal{A}_n^T \to 0$ strongly.

Since T is power bounded, $M := \sup_{n \geq 0} \|T^n\| < \infty$. We shall use the following two elementary formulas:

$$\mathcal{A}_{nk}^T = k^{-1}(\mathcal{A}_n^T + T^n \circ \mathcal{A}_n^T + T^{2n} \circ \mathcal{A}_n^T + \cdots + T^{(k-1)n} \circ \mathcal{A}_n^T) \tag{2.12}$$

and

$$\mathcal{A}_{j+i}^T - \mathcal{A}_j^T = (j+i)^{-1}(T^j + T^{j+1} + \cdots + T^{j+i-1}) - i(j+i)^{-1}\mathcal{A}_j^T\ . \tag{2.13}$$

Let $x \in E$ and $\varepsilon > 0$. Since

$$\lim_{n \to \infty} \|(I_E - P_u) \circ \mathcal{A}_n^T\, x\| = 0\,,$$

there exists n_ε such that $\mathrm{dist}(\mathcal{A}_{n_\varepsilon}^T x, B_u) \leq (3M)^{-1}\varepsilon$. Then there exist $c_\varepsilon \in \mathbb{R}_+$ and $w \in [-c_\varepsilon u, c_\varepsilon u]$ satisfying

$$\|\mathcal{A}_{n_\varepsilon}^T x - w\| \leq (2M)^{-1}\varepsilon\,.$$

Then, for any $l \geq 0$,

$$\|T^l \circ \mathcal{A}_{n_\varepsilon}^T\, x - T^l\, w\| \leq \|T^l\| \cdot \|\mathcal{A}_{n_\varepsilon}^T x - w\| \leq M \cdot \|\mathcal{A}_{n_\varepsilon}^T x - w\| \leq 2^{-1}\varepsilon\ . \tag{2.14}$$

$T[-u, u] \subseteq [-u, u]$ implies $T^l w \in [-c_\varepsilon u, c_\varepsilon u]$ for all l. Combining (2.12) and (2.14), we obtain

$$\mathrm{dist}(\mathcal{A}_{n_\varepsilon k}^T x, [-c_\varepsilon u, c_\varepsilon u]) \leq 2^{-1}\varepsilon \quad (\forall k \in \mathbb{N})\ . \tag{2.15}$$

By (2.13), there exists $k_\varepsilon \in \mathbb{N}$ satisfying

$$\|\mathcal{A}_{n_\varepsilon k+i}^T x - \mathcal{A}_{n_\varepsilon k}^T x\| \leq 2^{-1}\varepsilon \quad (\forall k \geq k_\varepsilon,\ i = 1, 2, \ldots, n_\varepsilon). \tag{2.16}$$

From (2.15) and (2.16), it follows that

$$\mathrm{dist}(\mathcal{A}_p^T x, [-c_\varepsilon u, c_\varepsilon u]) \leq \varepsilon \quad (\forall p \geq n_\varepsilon k_\varepsilon)\ . \tag{2.17}$$

By (2.17), the sequence $(\mathcal{A}_n^T x)_{n=0}^\infty$ is almost order bounded. Since every almost order bounded subset of a Banach lattice with order continuous norm is weakly precompact (cf. [101, Lemma 3.2]), $\{\mathcal{A}_n^T x\}_{n=0}^\infty$ has a weak cluster point and then, by the Eberlein theorem, the sequence $(\mathcal{A}_n^T x)_{n=1}^\infty$ is norm convergent for any $x \in E$. Thus T is mean ergodic. □

Remark that, from the proof above, we can see even more, namely, that $\mathrm{Fix}(T) \subseteq B_u$. Indeed, if $x \in \mathrm{Fix}(T)$, then

$$P_u^d x = (I_E - P_u)x = (I_E - P_u) \circ \mathcal{A}_n^T\, x \to 0.$$

So $P_u^d x = 0$, and hence $x \in B_u$.

2.2.6 Since order intervals in any KB-space are weakly compact, the theorem is true if we replace a weakly compact subset W of E by an order interval $[-g, g]$ for any $g \in E_+$. In this case, we have even more, that the fixed space $\mathrm{Fix}(T)$ of T is finite-dimensional, and this is what the next theorem shows (see [8]).

Theorem 2.2.5 (Alpay–Binhadjah–Emel'yanov–Ercan). *Let E be a KB-space, let T be a positive power bounded operator in E, let $g \in E_+$, and let $\eta \in \mathbb{R}$, $0 \le \eta < 1$, be such that*

$$\lim_{n\to\infty} \mathrm{dist}(\mathcal{A}_n^T x, [-g, g] + \eta B_E) = 0 \tag{2.18}$$

for any $x \in B_E$. Then T is mean ergodic and $\mathrm{Fix}(T)$ *is finite-dimensional.*

Proof. The mean ergodicity of T follows from Theorem 2.2.4.

Let us denote by C the order ideal generated by the set

$$\{x \in E_+ : \|\mathcal{A}_n^T x\| \to 0\}.$$

Then, for any $c \in C$, $\|\mathcal{A}_n^T c\| \to 0$. By the power boundedness of T, $\|\mathcal{A}_n^T x\| \to 0$ for any $x \in \overline{C}$, and hence the norm closure $\overline{C}$ of C coincides with C. Since any norm closed ideal in a Banach lattice with order continuous norm is a band [85, Cor.2.4.4], C is a band, and every band in E is a projection band, $E = C \oplus C^d$. Obviously, C is T-invariant. Denote by P_C the band projection $P_C : E \to C$, and by P_{C^d} the band projection $P_{C^d} : E \to C^d$. Let $T_1 := P_{C^d} \circ T$, then $0 \le T_1 \le T$, and the band C^d is T_1-invariant. The operator T_1 is power bounded, and

$$\lim_{n\to\infty} \mathrm{dist}(\mathcal{A}_n^{T_1} x, [-g, g] + \eta B_E) = 0 \quad (\forall x \in B_E).$$

Thus T_1 satisfies all conditions of Theorem 2.2.4, then, by this theorem, T_1 is mean ergodic. Consider the mean ergodic projections $P_T, P_{T_1} : E \to E$ defined as

$$P_T x = \lim_{n\to\infty} \mathcal{A}_n^T x, \quad P_{T_1} x = \lim_{n\to\infty} \mathcal{A}_n^{T_1} x \quad (\forall x \in E).$$

By Theorem 1.1.9, $\mathrm{Fix}(T) = P_T(E)$ and $\mathrm{Fix}(T_1) = P_{T_1}(E)$. Obviously

$$P_T, P_{T_1} \ge 0 \quad \text{and} \quad \mathrm{Fix}(T_1) \subseteq C^d.$$

Now we show that P_{T_1} is *strictly positive* in C^d in the sense that

$$x \in C^d_+, x \neq 0 \quad \Rightarrow \quad P_{T_1}x \neq 0.$$

Since C is T-invariant, we obtain by induction, that $P_{C^d} \circ T^n = P_{C^d} \circ T_1^n$ for all $n \geq 0$. Then $P_{C^d} \circ \mathcal{A}_n^T = P_{C^d} \circ \mathcal{A}_n^{T_1}$ for all $n \geq 0$, and hence

$$P_{C^d} \circ P_T = P_{C^d} \circ P_{T_1}. \tag{2.19}$$

Let $x \in C^d_+$, $x \neq 0$, then, by the construction of C, $P_T x \neq 0$ and $P_{C^d} \circ P_T x \neq 0$ since $P_T x \in \mathrm{Fix}(T)$. Then, by (2.19), $P_{C^d} \circ P_{T_1} x \neq 0$, and hence $P_{T_1} x \neq 0$, and so P_{T_1} is strictly positive in C^d. By [110, Thm.III.11.5], $\mathrm{Fix}(T_1)$ is a Banach sublattice in C^d and hence in E.

As it was shown in the proof of Theorem 2.2.4, there is a positive T_1-fixed vector u_1 of a maximal support and $(Id_{C^d} - P_{u_1}) \circ \mathcal{A}_n^{T_1} \to 0$ strongly as $n \to \infty$. Hence

$$\lim_{n \to \infty} \mathrm{dist}(\mathcal{A}_n^{T_1} x, [-P_{u_1} g, P_{u_1} g] + \eta B_E) = 0 \quad (\forall x \in E, \|x\| \leq 1). \tag{2.20}$$

Assume $\dim \mathrm{Fix}(T_1) = \infty$ then, by Judin's theorem (cf., [5, Exer. 13, p. 46]), there exists a sequence $(x_i)_{i=1}^\infty \subseteq \mathrm{Fix}(T_1)_+$ such that $x_i \wedge_{\mathrm{Fix}(T_1)} x_j = 0$ for all $i \neq j$. Hence $x_i \wedge x_j = 0$ for all $i \neq j$ since $\mathrm{Fix}(T_1)$ is a sublattice in E. We may assume $\|x_i\| = 1$. Set $y_i = P_{u_1} g \wedge x_i$ for any i. From (2.20), we obtain

$$\begin{aligned} \|y_i\| &= \|P_{u_1} g \wedge x_i\| \\ &= \|x_i - (x_i - P_{u_1} g)_+\| \\ &\geq 1 - \eta \\ &> 0 \end{aligned}$$

for all i. On the other hand, $(y_i)_{i=1}^\infty$ is an order bounded (by the element $P_{u_1} g$) disjoint sequence in E, so the order continuity of the norm in E implies $\|y_i\| \to 0$ due to [85, Thm.2.4.2], which contradicts the inequality above. Hence $\mathrm{Fix}(T_1)$ is finite-dimensional.

Now we show that $\mathrm{Fix}(T) \subseteq P_T(\mathrm{Fix}(T_1))$. From this, it will follow that

$$\dim(\mathrm{Fix}(T)) \leq \dim(\mathrm{Fix}(T_1)) < \infty,$$

which is required.

Indeed, let $f \in \mathrm{Fix}(T)$, then

$$\begin{aligned} f &= P_C f + P_{C^d} f \\ &= Tf \\ &= T \circ P_C f + T \circ P_{C^d} f, \end{aligned}$$

and

$$\begin{aligned} P_{C^d} f &= P_{C^d} \circ T \circ P_C\, f + P_{C^d} \circ T \circ P_{C^d}\, f \\ &= P_{C^d} \circ T \circ P_{C^d}\, f \\ &= T_1 \circ P_{C^d}\, f, \end{aligned}$$

since C is T-invariant. Hence $P_{C^d} f \in \mathrm{Fix}(T_1)$. To finish the proof of the theorem, it is enough to show that $f = P_T(P_{C^d} f)$. It follows directly from

$$\begin{aligned} f &= \mathcal{A}_n^T f \\ &= \mathcal{A}_n^T(P_C f) + \mathcal{A}_n^T(P_{C^d} f) \\ &\to P_T(P_{C^d} f) \quad (n \to \infty). \end{aligned}$$

□

Remark that any mean ergodic positive operator T, such that $\dim(\mathrm{Fix}(T)) < \infty$, satisfies the condition (2.18) for some $g \in E_+$ and $\eta \in \mathbb{R}$, $0 \le \eta < 1$. Moreover, η can be taken arbitrary small.

Example 1.3.8 shows that the condition that E is a KB-space cannot be omitted in Theorem 2.2.4. Even for Banach lattices with order continuous norm, this result can fail. Indeed, for $\alpha \neq 0$, the operator T_α constructed in Example 1.3.8 satisfies

$$\lim_{n\to\infty} \mathrm{dist}(\mathcal{A}_n^{T_\alpha} x, [-e_1, e_1] + \alpha B_{c_0}) = 0 \quad (\forall\ x \in B_{c_0}).$$

On the other hand, the sequence $(\mathcal{A}_n^{T_\alpha} e_1)_{n=1}^\infty$ does not converge to any element of c_0. Hence T_α is not mean ergodic.

2.2.7 Now we prove the following result, which was obtained by Räbiger [101, Thm.5.3] under some additional conditions.

Theorem 2.2.6 (Räbiger). *Let T be a positive operator in a KB-space E. Moreover, let $C := [-z, z] + \eta \cdot B_E$ be a constrictor of T, where $z \in E_+$ and $0 \le \eta < 1$. Then T is asymptotically periodic (see the definition in* **1.1.2***).*

Proof. Since the operator T satisfies the condition (2.18), by Theorem 2.2.5, T is mean ergodic. By Theorem 2.1.8, T has a constrictor $[-y, y]$, where $y \in E_+$, $Ty = y$. Since any order interval in a Banach lattice with order continuous norm is weakly compact, T is weakly almost periodic.

Show that the subspace $E_r(T)$ of reversible vectors of T is finite-dimensional, and $T|_{E_r(T)}$ is periodic. The Jacobs–Deleeuw–Glicksberg projection

$$P_T : E \to E_r(T)$$

is, obviously, positive. By [110, II.11.5], $E_r(T) = P_T(E)$ is a Banach lattice with respect to the order induced by E and with a suitable equivalent norm. Since $[-y, y] \in \mathrm{Constr}(T)$ and $P_T \in \text{wo-cl}\{T^n : n \in \mathbb{N}\}$, we have

$$B_{E_r(T)} \subseteq P_T(B_E) \subseteq [-y, y].$$

Applying P_T again, it follows that $B_{E_r(T)}$ is order bounded in $E_r(T)$. Thus $E_r(T)$ is lattice isomorphic to an AM-space with a strong unit, and hence, by the Kakutani–Krein representation theorem, $E_r(T) = C(K)$ for some compact Hausdorff space K. On the other hand, $E_r(T)$ is reflexive (since $B_{E_r(T)}$ is weakly compact) and, by Grothendieck's theorem (cf. [110, II.9.9, Cor.2]) $\dim(E_r(T)) < \infty$. Thus $T|_{E_r(T)}$ is a positive doubly power bounded operator in a finite-dimensional Banach lattice $E_r(T)$. By Theorem 1.1.15, $(T|_{E_r(T)})^{-1} \geq 0$ and hence $T|_{E_r(T)}$ is a lattice isomorphism in $E_r(T)$, but any lattice isomorphism in a finite-dimensional Banach lattice is periodic.

Now to finish the proof, it is enough to show that

$$E_{fl}(T) = \{x \in X : \lim_{n\to\infty} \|T^n x\| = 0\} .$$

Let F be the band in E generated by y. Then F has an order continuous norm and is a range of a positive projection $P : E \to F$. Moreover, there is a strictly positive $y' \in F^*$ (cf. [78, 1.b.15]) and $\psi := P^*(y')$ is a positive extension of y' to E.

Now, we apply the ultra-filter technique from **1.3.16** and **1.3.17**. Let $\mathcal{U}$ be a free ultra-filter on $\mathbb{N}$ and let $E_{\mathcal{U}}$ be the ultra-power of E with respect to $\mathcal{U}$. Define $\hat{T} : E_{\mathcal{U}} \to E_{\mathcal{U}}$ as in (1.48). Let $\hat{y} = \tilde{y} + E_0$, then $S(B_E) \subseteq [-\hat{y}, \hat{y}]$. Denote $(E_{\mathcal{U}})_{\hat{y}}$ the principal ideal generated by $\hat{y}$ in $E_{\mathcal{U}}$, then

$$\hat{T}_1 : E \to (E_{\mathcal{U}})_{\hat{y}}$$

is continuous. Thus $\hat{T} = i_{\hat{y}} \circ \hat{T}_1$ admits a factorization through $(E_{\mathcal{U}})_{\hat{y}}$, where

$$i_{\hat{y}} : (E_{\mathcal{U}})_{\hat{y}} \to E_{\mathcal{U}}$$

is the canonical injection. By means of

$$\langle \psi' , (x_n)_{n=1}^{\infty} + E_0 \rangle := \lim_{n\to\infty} \langle \psi , x_n \rangle ,$$

the linear functional ψ induces a functional $\psi' \in (E_{\mathcal{U}})^*$. Consider the L^1-space $(E_{\mathcal{U}}, \psi')$ associated to $E_{\mathcal{U}}$, which is a completion of $E_{\mathcal{U}}$ with respect to the norm given by ψ', and let

$$j_{\psi'} : E_{\mathcal{U}} \to (E_{\mathcal{U}}, \psi')$$

be the canonical map. We obtain the following diagrams:

$$E \xrightarrow{\hat{T}} E_{\mathcal{U}} \xrightarrow{j_{\psi'}} (E_{\mathcal{U}}, \psi');$$
$$E \xrightarrow{\hat{T}_1} (E_{\mathcal{U}})_{\hat{y}} \xrightarrow{i_{\hat{y}}} E_{\mathcal{U}} .$$

Since $i_{\hat{y}}$ and $j_{\psi'}$ are positive operators, and order intervals in the L^1-space $(E_{\mathcal{U}}, \psi')$ are weakly compact, $R := j_{\psi'} \circ i_{\hat{y}}$ maps the unit ball $[-\hat{y}, \hat{y}]$ of $(E_{\mathcal{U}})_{\hat{y}}$ into a weakly compact set, i.e., R is weakly compact. Since $(E_{\mathcal{U}})_{\hat{y}}$ is lattice isomorphic to a

$C(K)$-space, a result of Grothendieck (cf. [110, II.9.7, II.9.9]) implies that R maps weakly compact sets into norm compact sets. Thus $j_{\psi'} \circ \hat{T} = R \circ \hat{T}_1$ also maps weakly compact sets into norm compact sets.

Now let $x \in E_{fl}(T)$, $\|x\| \leq 1$. There is a subsequence $(T^{m_k}\, x)_{k=1}^{\infty}$ of $(T^m x)_{m=1}^{\infty}$ weakly convergent to zero. Thus $(j_{\psi'} \circ \hat{T} \circ T^{m_k}\, x)_{k=1}^{\infty}$ converges to zero in the norm, i.e.,

$$\begin{aligned} \lim_{\mathcal{U}} \lim_{k\to\infty} \langle \psi\, , |T^{n+m_k} x| \rangle &= \lim_{k\to\infty} \langle \psi'\, , |\hat{T} \circ T^{m_k} x| \rangle \\ &= \lim_{k\to\infty} \|j_{\psi'} \circ \hat{T} \circ T^{m_k}\, x\| \\ &= 0. \end{aligned}$$

In particular, there is a sequence $(n_k)_{n=1}^{\infty}$ of naturals such that

$$\lim_{k\to\infty} \langle \psi\, , |T^{r_k} x| \rangle = 0 \qquad (r_k := n_k + m_k). \tag{2.21}$$

Since $[-y, y] \in \mathrm{Constr}(T)$, there is a decomposition

$$|T^{r_k} x| = a_k + b_k \quad (k \in \mathbb{N})\,,$$

such that $a_k \in [-y, y] \cap E_+$, $b_k \in E_+$, and $\lim_{k\to\infty} \|b_k\| = 0$. The sequence $(a_k)_{k=1}^{\infty}$ is order bounded in F and (2.21) implies

$$\begin{aligned} \lim_{k\to\infty} \langle y'\, , a_k \rangle &= \lim_{k\to\infty} \langle P^*(y')\, , a_k \rangle \\ &= \lim_{k\to\infty} \langle \psi\, , a_k \rangle \\ &= 0\,. \end{aligned}$$

By Lemma 2.2.1, we obtain $\lim_{k\to\infty} \|a_k\| = 0$, thus

$$\lim_{k\to\infty} \|T^{r_k} x\| = \lim_{k\to\infty} \|a_k + b_k\| = 0.$$

Since T is power bounded, this implies

$$\lim_{n\to\infty} \|T^n x\| = 0\,,$$

and the proof is finished. □

Related Results and Notes

2.2.8 Prove the following easy facts about positive operators in Banach lattices.

Exercise 2.2.7. Show that every positive linear operator T from a Banach lattice E to another Banach lattice F is norm-continuous. In particular, every positive linear functional ψ on E belongs to E^*.

Exercise 2.2.8. Let $T : E \to F$ be a positive invertible linear operator between Banach lattices E and F. Show that the following conditions are equivalent:

(i) T^{-1} is positive;

(ii) T is a lattice isomorphism.

2.2.9 Let us say several words about the history of Theorems 2.2.2 and 2.2.3. In the last two decades, there were many results concerning spectral and asymptotic properties of dominated operators, such as strong and uniform ergodicity, strong and uniform stability, almost periodicity, quasi-compactness or certain inclusion relations among the peripheral spectra (see [9], [12], [19], [84], [92], [99], [100], [101], [102], [103], etc.). In [100, Thm. 3.9], it was shown that if T is a positive operator on a Banach lattice E with order continuous norm such that the powers T^n converge strongly to a projection P_T of finite rank, then, for each operator S on E such that $0 \le S \le T$, the powers S^n are also strongly convergent. The same conclusion holds if one requires instead of the finite rank condition on P_T that the spectrum $\sigma(T)$ of T does not contain the whole unit circle (see [102, Cor. 4.3]). Analogous statements hold for the inheritance of almost periodicity under domination (see [100, Prop. 3.10], [102, Thm. 4.2]). What kind of additional conditions on the dominating operator T (or on the dominating C_0-semigroup $\mathcal{T}$) are needed to make the dominated semigroup $\mathcal{S}$ almost periodic or strongly stable? Theorems 2.2.2 and 2.2.3 give the complete answer to this question.

Also we remark that the second part of the proof of Theorem 2.2.2 shows that the strong continuity of $\mathcal{T}$ is not necessary, and that it is sufficient to require only asymptotic compactness of $\mathcal{T}$ instead of the almost periodicity of $\mathcal{T}$.

Here we present (see [32]) a short, but indirect, proof of Theorem 2.2.3 founded on a result of the paper [102].

An alternative proof of Theorem 2.2.3. Theorem 2.2.2 implies that $\mathcal{S}$ is almost periodic. Since $\mathcal{S}$ is asymptotically dominated by $\mathcal{T}$ and $\mathcal{T}$ is strongly stable, the corresponding Jacobs–Deleeuw–Glicksberg projections $Q_{\mathcal{S}}$ and $Q_{\mathcal{T}}$ satisfy

$$0 \le S_t \circ Q_{\mathcal{S}} \le T_t \circ Q_{\mathcal{T}} = Q_{\mathcal{T}} \quad (t \in J).$$

By [102, Thm.1.4], the following inclusions for the spectra hold:

$$\sigma(S_t \circ Q_{\mathcal{S}}) \cap \Gamma \subseteq \sigma(Q_{\mathcal{T}}) \cap \Gamma \subseteq \{1\} \quad (t \in J).$$

Now the Jacobs–Deleeuw–Glicksberg theorem yields

$$S_t|_{E_r(\mathcal{S})} = I_{E_r(\mathcal{S})}$$

and $\lim\limits_{t\to\infty} S_t x = 0$ for all $x \in E_0(\mathcal{S})$. Hence $\mathcal{S}$ is strongly stable. □

In the paper [98, Prop. 3.1 and 3.4], it was shown that in order to have inheritance of almost periodicity and stability, as formulated in Theorems 2.2.2, 2.2.3 respectively, the order continuity of the norm of E is necessary, at least for Banach lattices which contain a topological orthogonal system or which are σ-Dedekind complete.

The following examples [32] show that positivity of $\mathcal{S}$ cannot be omitted in Theorems 2.2.2, 2.2.3. We point out that in the second example the operators belonging to the semigroup $\mathcal{S}$ are even regular.

a) Consider the operator S in $L^p(\Gamma),\ 1 \le p < \infty$, defined by

$$Sf(z) = zf(z),\ f \in L^p(\Gamma),\ z \in \Gamma.$$

Then $\mathcal{S} = (S^n)_{n=1}^{\infty}$ is dominated by the trivial semigroup $\mathcal{T} = (Id)_{n=0}^{\infty}$ in the following sense:

$$|S^n f| = |f|, \quad f \in L^p(\Gamma).$$

Note that S is an isometry.

Exercise 2.2.9. Show that S has no eigenvalue. Thus, by the Jacobs–Deleeuw–Glicksberg theorem, $\mathcal{S}$ cannot be almost periodic, whereas $\mathcal{T}$ is even strongly stable.

b) Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. Define S on $L^p(\Gamma),\ 1 \le p < \infty$, by

$$(Sf)(z) = h(z) f(e^{i\alpha\pi} z)\,,$$

where $h(z) = 1$ if $\mathrm{Re}(z) \ge 0$ and $h(z) = -1$ if $\mathrm{Re}(z) < 0$. The modulus of the operator S exists and is given by $(|S|\, f)(z) = f(e^{i\alpha\pi} z)$.

Exercise 2.2.10. Show that the semigroup $\mathcal{T} = (|S|^n)_{n=0}^{\infty}$ is almost periodic.

The semigroup $\mathcal{S} = (S^n)_{n=0}^{\infty}$ is dominated by $\mathcal{T}$ in the following sense:

$$|S^n f| \le |S|^n |f|, \quad f \in L^p(\Gamma).$$

Exercise 2.2.11. Show that $\{S^n \mathbb{1} :\ n \in \mathbb{N}\}$ is not relatively compact in $L^p(\Gamma)$, where $\mathbb{1}$ denotes the constant function equal to 1 on Γ.

Hence $\mathcal{S}$ is not almost periodic.

2.2.10 In a Banach lattice with order continuous norm, it follows from the mean ergodicity of a positive power bounded operator T that T^m is mean ergodic for any $m \in \mathbb{N}$ [24] (see also Theorem 2.1.14). In general, it is not true, even for a Koopman operator on a Banach lattice $C(K)$, for appropriate compact Hausdorff space K (see Sine's paper [119]). The idea of the correspondent example in [119] is quite technical and cannot be used directly in any $C(K)$.

Open Problem 2.2.12. *Let K be an infinite compact Hausdorff space. Is there a positive power bounded mean ergodic operator T in $C(K)$ such that T^m is not mean ergodic for some $m \in \mathbb{N}$?*

Another interesting question is related to the Sine mean ergodic theorem (see Theorem 1.1.11). What is its form for the case of positive operators in Banach lattices?

Open Problem 2.2.13. *Let T be a positive Cesàro bounded operator in a Banach lattice which satisfies* (1.7) *such that positive fixed vectors T separate positive fixed vectors of T^*. Is T mean ergodic*?

We finish with the following problem on positive operators in a Banach lattice. It is easy to see, using the trivial fact that the weak convergence of a positive sequence in an AL-space implies the strong convergence, that any power bounded positive operator T on an AL-space E, such that 0 belongs to the weak-closure of an orbit $\{T^n x\}_{n=1}^{\infty}$ for each $x \in E$, satisfies a formally more strong condition

$$\text{w-}\lim_{n\to\infty} T^n x = 0 \quad (\forall x \in E).$$

It is a well-known fact that, in general, this is not true for power bounded operators in Banach spaces. However, if we assume $T \in \mathcal{L}(X)$ to be almost periodic then, by Theorem 1.1.4, it follows from $0 \in \text{w-cl}\{T^n x\}_{n=1}^{\infty}$ for each $x \in X$ that $\lim_{n\to\infty} \|T^n x\| = 0$ for all $x \in X$. It is an interesting and still open question to investigate the case of positive operators (or positive C_0-semigroups) in Banach lattices.

Open Problem 2.2.14. *Let $\mathcal{T}$ be a one-parameter bounded positive semigroup in a Banach lattice E which satisfies $0 \in \text{w-cl}\{T^n x\}_{n=1}^{\infty}$ for each $x \in E$. Does*

$$\text{w-}\lim_{n\to\infty} T^n x = 0$$

hold for all $x \in E$?

2.2.11 Let us say some words about renorming of a Banach lattice in connection with asymptotic behavior of a positive operator acting in it. First, if we have a power bounded positive operator T in a Banach lattice E, then we can easily renorm E,

$$\|x\|_T := \sup\{\|T^n x\| : n \geq 0\} \qquad (\forall x \in E)$$

to make T contractive: $\|x\|_T \geq \|Tx\|_T$ for all $x \in E$. This idea works also for a doubly power bounded positive operator T such that T^{-1} is positive. In this case, it can be very easily shown that T is a lattice isomorphism, and we can construct the equivalent norm on E under which T becomes an isometry:

$$\|x\|_T := \sup\{\|T^n x\| : n \in \mathbb{Z}\} \qquad (\forall x \in E).$$

If T is positive and doubly power bounded, the similar procedure, in general, does not lead us to a positive isometry (because of $T^{-1} \not\geq 0$ in general), as the following example [29] shows.

Example 2.2.15. Given $\omega \notin \mathbb{R}$, take $\Omega = \mathbb{R} \cup \{\omega\}$, and let a measure μ on the Borel algebra $\mathcal{B} = \mathcal{B}(\Omega)$ be defined as the Lebesgue measure on $\mathcal{B}(\mathbb{R})$, and let $\mu(\{\omega\}) = 1$.

Then, for any $\varepsilon > 0$, there exists a positive operator in $L^1(\Omega, \mathcal{B}, \mu)$ with non-positive inverse that satisfies $\sup_{n\in\mathbb{Z}} \|T^n\| \leq 1 + \varepsilon$.

Proof. Let $\varepsilon > 0$. Consider a measure preserving automorphism S in Ω defined by $S(\omega) = \omega$ and $S(t) = t - 1$, whenever $t \in \mathbb{R}$; and define a positive operator T in $L^1(\Omega, \mathcal{B}, \mu)$ by

$$Tf := f \circ S + \varepsilon \cdot \Big[\int_0^1 f d\mu \Big] \cdot \chi_{\{\omega\}},$$

where, as usual, χ_A is the indicator function of a subset A. Then it is easy to see that

$$\begin{aligned} T^n f &= f \circ S^n + \varepsilon \cdot \Big[\sum_{i=1}^{n} \int_0^1 f \circ S^{i-1} d\mu \Big] \cdot \chi_{\{\omega\}} \\ &= f \circ S^n + \varepsilon \cdot \Big[\sum_{i=1}^{n} \int_{n-i}^{n-i+1} f \circ S^{n-1} d\mu \Big] \cdot \chi_{\{\omega\}} \\ &= f \circ S^n + \varepsilon \cdot \Big[\int_0^n f \circ S^{n-1} d\mu \Big] \cdot \chi_{\{\omega\}} \end{aligned}$$

for all $n \in \mathbb{Z}_+$, and $f \in L^1(\Omega, \mathcal{B}, \mu)$. A similar computation shows that

$$T^n f = f \circ S^n - \varepsilon \cdot \Big[\int_{-n}^0 f \circ S^{n-1} d\mu \Big] \cdot \chi_{\{\omega\}}$$

for all $n \in \mathbb{Z} \backslash \mathbb{Z}_+$; and $f \in L^1(\Omega, \mathcal{B}, \mu)$, in particular, T^{-1} is not positive. Moreover,

$$\begin{aligned} \|T^n f\| &\leq \|f \circ S^n\| + \varepsilon \cdot \Big[\int_{-\infty}^{\infty} |f \circ S^{n-1}| d\mu \Big] \cdot \|\chi_{\{\omega\}}\| \\ &= (1 + \varepsilon)\|f\| \end{aligned}$$

for all $n \in \mathbb{Z}$; and $f \in L_1(\Omega, \mathcal{B}, \mu)$, which provides the required property $\sup\{\|T^n\| : n \in \mathbb{Z}\} \leq 1 + \varepsilon$. □

This result was extended in [7] to any AL-space. Also, by the duality, if an AM-space E has a predual, then there is a positive doubly power bounded operator S in E such that S^{-1} is not positive. We do not know if the similar example can be constructed for any infinite-dimensional Banach lattice, or not. But, in the finite-dimensional case, as Theorem 1.1.15 says, it is impossible.

We continue with the following theorem of Abramovich [1]. Indeed, in the paper [1], a slightly more general result was proved, but we restrict our attention only to the following special case.

Theorem 2.2.16 (Abramovich). *Any surjective positive isometry in a Banach lattice has a positive inverse.*

Proof. Let T be a surjective positive isometry in a Banach lattice E. We have to show that $T(E_+) = E_+$. Assume that it is not true. Then there exists $y \in E_+$ such that $y \notin T(E_+)$. Let $\|y\| = 1$. Take $x \in E$ such that $Tx = y$. Then $\|x\| = 1$ and $x \notin E_+$. Consequently, $x_- > 0$ and $x_+ \neq 0$, since if $x_+ = 0$, then

$$y = Tx = -T(x_-) < 0\,,$$

which is impossible. Set $y_1 = T(x_+)$, $y_2 = T(x_-)$. Then $y_1 > 0$, $y_2 > 0$, and $y = y_1 - y_2 > 0$. Moreover, $T(x_+ - x_-) = y_1 + y_2$, and

$$\|y_1 + y_2\| = \|x_+ + x_-\| = \|x_+ - x_-\| = \|x\| = 1. \tag{2.22}$$

We shall prove by induction that for any $k \in \mathbb{N}$,

$$\|x_+ + kx_-\| = 1. \tag{2.23}$$

For $k = 1$, (2.23) follows from (2.22) directly. Let (2.23) be true for k; we will prove it for $k+1$.

Let us consider the element $x_+ - (k+1)x_-$. Then

$$T(x_+ - (k+1)x_-) = y_1 - (k+1)y_2,$$

and

$$-(y_1 - ky_2) \leq y_1 - (k+1)y_2 \leq y_1 + ky_2.$$

Then $|y_1 - (k+1)y_2| \leq y_1 + ky_2$ and

$$\begin{aligned} \|y_1 - (k+1)y_2\| &\leq \|y_1 + ky_2\| \\ &= \|T(x_+ + kx_-)\| \\ &= \|x_+ + kx_-\| \\ &= 1. \end{aligned}$$

Thus

$$\|x_+ - (k+1)x_-\| = \|T^{-1}(y_1 - (k+1)y_2\| \leq 1,$$

and

$$\|x_+ + (k+1)x_-\| = \|x_+ - (k+1)x_-\| \leq 1.$$

Since $\|x_+ + (k+1)x_-\| \geq 1$, we get

$$\|x_+ + (k+1)x_-\| = 1\,,$$

and (2.23) is proved for all $k \in \mathbb{N}$. Then $x_- = 0$, which contradicts our assumption. Then $T(E_+) = E_+$, which is required. $\square$

2.2.12 The following spectral characterization of the uniform mean ergodicity of positive operators in a Banach lattice [59, Thm.5] is stronger than the similar result (Theorem 1.1.34) for operators in a Banach space.

Theorem 2.2.17 (Karlin). *A positive operator T in a Banach space is uniformly mean ergodic iff $r(T) \leq 1$ and 1 is a pole of the resolvent of T of order at most 1.* □

There are several interesting results about inheritance of uniform mean ergodicity and uniform stability. We refer to Räbiger [99] and mention here some of these results. The next three theorems are mainly due to Caselles [19, Cor. 4.6] (cf., also, [99, Thm. 2.2]) and Räbiger [99, Cor. 2.6 and Thm. 3.4]).

Theorem 2.2.18 (Caselles). *Let E be a Banach lattice and let $S, T \in \mathcal{L}(E)$ satisfy $0 \leq S \leq T$. If T is uniformly mean ergodic with the ergodic projection P_T of finite rank, then S is uniformly mean ergodic and* $\mathrm{rank}(P_S) \leq \mathrm{rank}(P_T)$. □

Theorem 2.2.19 (Räbiger). *Let E be a Banach lattice and let $S, T \in \mathcal{L}(E)$ satisfy $0 \leq S \leq T$. If T is uniformly stable with the ergodic projection P_T of finite rank, then S is uniformly stable and* $\mathrm{rank}(P_S) \leq \mathrm{rank}(P_T)$. □

Theorem 2.2.20 (Räbiger). *Let E be a Banach lattice and let $\mathcal{S}$ and $\mathcal{T}$ be positive C_0-semigroups in E such that S is dominated by $\mathcal{T}$. If $\mathcal{T}$ is uniformly mean ergodic and* $\mathrm{rank}(P_{\mathcal{T}}) < \infty$, *then $\mathcal{S}$ is uniformly mean ergodic and* $\mathrm{rank}(P_{\mathcal{S}}) \leq \mathrm{rank}(P_{\mathcal{T}})$. □

The uniformly stable version of Theorem 2.2.20 is due to Martinez and Mazon [84, Prop. 3.3] (cf. also [99, Thm. 3.6]).

Theorem 2.2.21 (Martinez–Mazon). *Let E be a Banach lattice and let $\mathcal{S}$ and $\mathcal{T}$ be positive C_0-semigroups in E such that S is dominated by $\mathcal{T}$. If $\mathcal{T}$ is uniformly stable and* $\mathrm{rank}(P_{\mathcal{T}}) < \infty$, *then $\mathcal{S}$ is uniformly stable and* $\mathrm{rank}(P_{\mathcal{S}}) \leq \mathrm{rank}(P_{\mathcal{T}})$. □

Remark that the inequalities on ranks in the theorems above are obvious by Proposition 2.1.3. The only non-trivial part is to prove that the dominated semigroup is uniformly mean ergodic. Theorems 2.2.18 and 2.2.19 are no longer true if P_T has infinite rank as the following exercise shows.

Exercise 2.2.22. Let $E = \ell_p$, $1 \leq p < \infty$, $T = I_E$, and $S \in \mathcal{L}(E)$ is defined as follows (cf. Example 2.1.24) :

$$S((x_n)_{n=1}^{\infty}) = \left(\frac{n}{n+1}\, x_n\right)_{n=1}^{\infty}.$$

Then $0 \leq S \leq T$ and T is, obviously, uniformly stable. Show that S is even not mean ergodic.

Exercise 2.2.23. Let P be a positive projection of finite rank in a Banach lattice E. Show that every $S \in \mathcal{L}(E)$ satisfying $0 \leq S \leq P$ is uniformly stable.

An operator T in a Banach space is called *uniformly asymptotically periodic* if there is a periodic operator G such that $\lim_{n\to\infty} \|T^n - G^n\| = 0$.

Exercise 2.2.24. Let E be a Banach lattice and let $S, T \in \mathcal{L}(E)$ satisfy $0 \leq S \leq T$. Assume that T is uniformly asymptotically periodic with the corresponding periodic operator P of finite rank. Show that S is uniformly asymptotically periodic and its periodic operator Q satisfies $\mathrm{rank}(Q) \leq \mathrm{rank}(P)$. Show that $\mathrm{aper}(T) = k \cdot \mathrm{aper}(S)$ for an appropriate $k \in \mathbb{N}$.

Hint: See the proof of [99, *Prop.*2.8]

2.2.13 In the following exercise, the reader should show that the order continuity of the norm in Theorems 2.2.2 and 2.2.3 cannot be omitted even if $\mathrm{rank}(\mathcal{T}) = 1$.

Exercise 2.2.25. Let $E = c$, and let $T, S \in \mathcal{L}(E)$ be defined as follows:

$$T((x_n)_{n=1}^{\infty}) = ((x_{n+1})_{n=1}^{\infty}) \ \ \& \ \ S((x_n)_{n=1}^{\infty}) = \left(\frac{n}{n+1}\, x_{n+1}\right)_{n=1}^{\infty}.$$

Show that T is strongly stable with $\mathrm{rank}(P_T) = 1$. Show that S is even not mean ergodic.

On the other hand, under some additional assumptions, Theorem 2.2.2 is still true for any Banach lattice. For example, the following result has been stated in [100, Lm.3.4].

Theorem 2.2.26 (Räbiger). *Let $S, T \in \mathcal{L}(E)$ be operators in a Banach lattice E such that $0 \leq S \leq T$. Let T be strongly stable and let $\mathrm{rank}(P_T) < \infty$. If S is almost periodic, then S is strongly stable and $\mathrm{rank}(P_S) \leq \mathrm{rank}(P_T)$.* □

It is an open question if in this theorem one can omit the finite rank condition on P_T.

2.2.14 There is an old and well developed spectral theory of positive operators in Banach lattices, we refer to [85, Ch.4] for it. We mention here without proofs only a few results. The first and, probably, the most famous one concerns positive compact operators.

Theorem 2.2.27 (Krein–Rutman). *Let T be a positive operator in a Banach lattice. If T^k is compact for some $k \in \mathbb{N}$, then the spectral radius $r(T)$ is an eigenvalue of T corresponding to a positive eigenvector. Moreover, $r(T)$ is a pole of $R_\lambda(T)$ of the maximal order on the spectral circle $\{\lambda \in \sigma(T) : |\lambda| = r(T)\}$.* □

Also, there are several results about the spectrum of a Riesz homomorphism.

Theorem 2.2.28 (Lotz–Scheffold). *Let T be a Riesz homomorphism in a Banach lattice. Then the spectrum $\sigma(T)$, point spectrum $\sigma_p(T)$, and approximative spectrum $\sigma_a(T)$ are cyclic subsets of $\mathbb{C}$.* □

Theorem 2.2.29 (Arendt–Schaefer–Wolff). *Let T be a Riesz homomorphism in a Banach lattice such that $\sigma(T) = \{1\}$. Then $T = I$.* □

Theorem 2.2.29 was stated in [113] and was generalized in [134] in the following way.

Theorem 2.2.30 (Zhang). *Let T be a positive operator in a Banach lattice such that 0 belongs to the unbounded component of $\mathbb{C} \setminus \sigma(T)$. Then, for every ε,*

$$0 < \varepsilon < r(T^{-1})^{-1},$$

there is $n = n(\varepsilon)$ satisfying

$$T^{n \cdot m} \geq (r(T^{-1})^{-1} - \varepsilon)^{n \cdot m} \cdot I \quad (\forall m \in \mathbb{N}). \qquad \square$$

It is interesting that the extension of Theorem 2.2.29 onto the class of positive operators is a hard and still open problem, namely:

Open Problem 2.2.31. *Let T be a positive operator in a Banach lattice such that $\sigma(T) = \{1\}$. Does $T \geq I$ hold?*

There is only one class of Banach lattices, in which this problem was solved (for the positive case), the finite-dimensional Banach lattices. In any class of infinite dimensional Banach lattices this problem is open. However, under some additional conditions on an operator this problem has the positive solution. The following result was obtained very recently in [26, Cor.1] by using the technique developed by Zhang [134].

Theorem 2.2.32 (Drnovšek). *Let T be a positive operator in a Banach lattice such that $\lim_{n\to\infty} n\|(T-I)^n\|^{1/n} = 0$. Then $T \geq I$.* $\square$

Another class of results concerns the so-called *zero-two law*. For details, we also send the reader to [85, Ch.4]. For example, the following theorem, which is due to Schaefer [111], is about positive contractions.

Theorem 2.2.33 (Schaefer). *Let T be a positive contraction in a Banach lattice E, then either $\lim_{n\to\infty} \|T^n - T^{n+1}\| = 0$ or*

$$\|T^n - T^{n+1}\|_r = 2 \quad (\forall n = 0, 1, \ldots),$$

where $\|S\|_r = \inf\{\|A\| : A \in \mathcal{L}(E),\ |Sx| \leq Ax\ \forall x \in E_+\}$ is the regular norm *of a regular operator $S \in \mathcal{L}(E)$.* $\square$

2.3 Positive semigroups and geometry of Banach lattices

It is well known (see Theorem 1.1.20) that every power bounded operator in a reflexive Banach space is mean ergodic. Is the converse true? This old problem of the theory of Banach spaces was suggested by Sucheston in 1975 in [126]. In 1986, Zaharopol [132] gave the positive answer to Sucheston's question for a σ-Dedekind complete Banach lattice. Zaharopol's result has been generalized in many directions in [27], [34], [33], [45], and [98]. Let us mention Räbiger's paper [98], in which there was constructed an example of non-mean ergodic operator T satisfying $0 \leq T \leq I$ in a Banach lattice which is σ-Dedekind complete or contains a topological orthogonal system and which does not have the order continuous norm. The original Sucheston's problem is still open. In this section, we consider only the Banach lattice setting and present some results from [8], [27], [33], [34], and [132]. Another question arises about geometrical properties of a Banach space or a Banach lattice, which can be characterized by the mean ergodicity of operators belonging to some special classes. We shall also discuss this question.

2.3.1 In order to describe our first result [34], we need the following notion. Let E be a Banach lattice and let $T \in \mathcal{L}(E)$. We call the operator T *power order bounded* if for every $x \in E_+$ there exists $z \in E_+$ such that $T^n([-x,x]) \subseteq [-z,z]$ for all $n \in \mathbb{N}$. We characterize the order continuity of the norm on a Banach lattice as follows.

Theorem 2.3.1 (Emel'yanov–Wolff). *Let E be a Banach lattice. Then the following assertions are equivalent*:

(i) *the norm on E is order continuous*;

(ii) *every power order bounded operator $T : E \to E$ is mean ergodic.*

Before we give the proof, let us recall the following two well-known facts. The first one is that any power order bounded operator is power bounded by the uniform boundedness principle. The second one is: let $h \in c_0$ satisfy

$$0 < h(n) \leq 1 \quad (\forall n \in \mathbb{N}),$$

then the multiplication operator S_h in ℓ^∞ given by

$$f \mapsto S_h(f) = h\, f$$

maps ℓ^∞ into c_0 and is compact, since the unit ball of ℓ^∞ (which coincides with the order interval $[-\mathbb{1}, \mathbb{1}]$) is mapped into the order interval $[-h, h]$, which is known to be compact. Here $\mathbb{1}$ denotes the constant function $n \mapsto 1$.

Proof of Theorem 2.3.1. (i) $\Rightarrow$ (ii): By the first remark before the proof, T is power bounded. Let $x \in E$ be arbitrary. Then there exists $u \geq 0$ such that $T^n x \in [-u, u]$ for all n. Since the order interval $[-u, u]$ is weakly compact, the assertion follows from Theorem 1.1.7.

(ii) $\Rightarrow$ (i): (I) Assume that the norm on E is not order continuous. Then by [85, Thm.2.4.2], there exists a disjoint order bounded sequence $(e_n)_{n=1}^\infty$ of E_+ which does not converge to 0 in norm. So without lost of generality we may assume that $\|e_n\| = 1$ and $e_n \leq u$ for some u and all n. By [110, Exer. 18.b, p. 147], there exists a disjoint normalized sequence $(\psi_n)_{n=1}^\infty$ in E_+^* such that $\psi_n(e_m) = 0$ for $m \neq n$ and $\psi_n(e_n) \geq 1/2$. We set

$$\varphi_n = \frac{\psi_n}{\psi_n(e_n)}\,.$$

Then $\|\varphi_n\| \leq 2$ and $\varphi_n(e_m) = \delta_{n,m}$.

(II) Since $\sum_{k=1}^n e_k = \sup(e_1, \ldots, e_n) \leq u$, the map $U : c_0 \to E$, given by

$$U(f) = \sum_{n=1}^\infty f(n)e_n,$$

is a well-defined topological lattice isomorphism into E (see [85, Lm.2.3.10]). Its range will be denoted by E_0. Define $V : E \to \ell^\infty$ by $V(x)(n) := \varphi_n(x)$. Then $\|V\| \leq 2$ and $V \circ U = Id$ on c_0. Moreover, $V(u) \geq V(e_n)$ for every n, hence $V(u) \geq 1\!\!1$, in particular, $V(u) \notin c_0$.

(III) Let h be as in the remark before the proof. Then $A = U \circ S_h \circ V$ is well defined since $S_h(\ell^\infty) \subseteq c_0$. Moreover, A is compact and positive as S_h is. Set $T = I_E - A$.

Claim:

$$T^r = I_E - U \circ \sum_{k=1}^r \binom{r}{k} (-1)^{k-1} S_h^k \circ V. \tag{2.24}$$

Proof: Since $S_h(\ell^\infty) \subseteq c_0$ and $V \circ U = Id$ on c_0, we obtain by the induction

$$(U \circ S_h \circ V)^r = U \circ S_h^r \circ V \quad (\forall r \geq 1).$$

This in turn yields

$$\begin{aligned} T^r &= (I_E - A)^r \\ &= I_E + \sum_{k=1}^r \binom{r}{k} (-1)^k (U \circ S_h \circ V)^k \\ &= I_E - U \circ \sum_{k=1}^r \binom{r}{k} (-1)^{k-1} S_h^k \circ V. \end{aligned}$$

(IV) Now define R_r in ℓ^∞ by

$$R_r = \sum_{k=1}^r \binom{r}{k} (-1)^{k-1} S_h^k\,.$$

Then

$$R_r = I_{\ell^\infty} - (I_{\ell^\infty} - S_h)^r \geq 0,$$

since $0 \le h \le 1\!\!1$ implies $0 \le I_{\ell^\infty} - S_h \le I_{\ell^\infty}$. This implies $U \circ R_r \circ V \ge 0$. For a function $f \in \ell^\infty$, denote its truncation at n by $f^{(n)}$ defined by

$$f^{(n)}(k) = \begin{cases} f(k) & k \le n \\ 0 & \text{else} \end{cases} .$$

Claim: T is power order bounded. More precisely: for $|y| \le x$ and all r, we have $|T^r y| \le x + 2\|x\|\, u$.

Proof: Equation (2.24) and the positivity of $U \circ R_r \circ V$ together imply

$$\begin{aligned} |T^r y| &\le |y| + U \circ R_r \circ V\,(|y|) \\ &\le x + U \circ R_r \circ V\,(x). \end{aligned}$$

Now

$$\begin{aligned} R_r \circ V\,(x) &= (I_{\ell^\infty} - (I_{\ell^\infty} - S_h)^r) \circ V\,(x) \\ &\le V\,(x) \\ &\le 2\|x\| \cdot 1\!\!1\,. \end{aligned}$$

Set $f = R_r \circ V\,(x)$. Then

$$U f^{(n)} \le 2\|x\| \sum_{k=1}^{n} e_k \le 2\|x\| \cdot u.$$

Now f is in c_0, and this implies $f = \lim_{n\to\infty} f^{(n)}$, hence

$$U \circ R_r \circ V\,(x) = U\,(f) \le 2\|x\| \cdot u.$$

The claim is proved.

(V) *Claim*: T is not mean ergodic.

Proof: Assume that T is mean ergodic. Then, by Theorem 1.1.9,

$$E = \ker(A) \oplus \overline{A(E)}\,.$$

Since U and S_h are injective, we obtain $E = \ker(V) \oplus E_0$. So, in particular, $u = v + w$, where $V(v) = 0$ and $V(u) = V(w) \in c_0$, a contradiction. □

2.3.2 We show that a Banach space, in which every power bounded operator is mean ergodic, does not contain c_0 (see [34]).

Theorem 2.3.2 (Emel'yanov–Wolff). *Let E be a Banach space in which every power bounded operator is mean ergodic. Then E does not contain a lattice isomorphic copy of c_0.*

Proof. (I) Assume that there exists a topological isomorphism of c_0 into E. Since c_0 and c are topologically isomorphic, there exists a topological isomorphism U of c into E. Its dual U^* maps E^* topologically onto $c^* = \ell^1(\mathbb{N}_0)$, where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Let $e_n \in \ell^1(\mathbb{N}_0)$ be the sequence defined by

$$e_n(f) = \begin{cases} f(n) & n \geq 1 \\ \lim_{k\to\infty} f(k) & n = 0 \end{cases}$$

for all $f \in c$. Since the operator U^* is open, there exists $M > 0$ and, to each $n \geq 1$, a linear functional $\psi_n \in E^*$ satisfying $\|\psi\| \leq M$ and $U^*\psi = e_n$.

(II) Define $V : E \to \ell^\infty$ by $V(x) = (\psi_n(x))_{n\geq 1}$ and let h be as in the proof of Theorem 2.3.1. Set

$$A = U \circ S_h \circ V\,.$$

Then $T = I - A$ is again well defined, and calculations similar to those in the proof of Theorem 2.3.1 show that T is power bounded.

(III) Assume that T is mean ergodic. Then we have $E = \ker(A) \oplus \overline{A(E)}$. Since U and S_h are injective, we obtain

$$E = \ker(V) \oplus U(c_0)\,.$$

So, in particular, $U(\mathbb{1}) =: u = v + w$, where $V(v) = 0$, hence

$$V(u) = V(w) \in c_0\,,$$

a contradiction to $V(u) = \mathbb{1}$. □

The proof of the existence of a power bounded operator which is not mean ergodic in a separable Banach space X containing c_0 is an easy consequence of the Sobczik theorem (see [23], p.71), that provides a bounded projection of X onto c_0.

2.3.3 Now we give a proof of the following result [132]. Actually, Zaharopol has proved the stronger assertion, that for reflexivity of a σ-Dedekind complete Banach lattice E it is necessary and sufficient that every positive power bounded operator in E is mean ergodic. We refer for this result to [132] and [98].

Theorem 2.3.3 (Zaharopol). *Let E be a σ-Dedekind complete Banach lattice. If every positive power bounded operator $T : E \to E$ is mean ergodic, then E is reflexive.*

Proof. By Theorem 2.3.1, the norm on E is order continuous. It is known that any Banach lattice which norm is order continuous is reflexive if and only if it contains neither a lattice isomorphic copy of ℓ^1, nor of c_0 (see [85, Thm.2.4.15]). By Theorem 2.3.2, E does not contain a copy of c_0.

Assume that there is a Banach sublattice E_0 of E which is lattice isomorphic to ℓ^1, with an ℓ^1-basis $(x_n)_{n=1}^\infty$ and the basis constant γ,

$$\|x\| \leq \gamma \sum_{n=1}^{\infty} |\alpha_n| \quad \left(\forall x = \sum_{n=1}^{\infty} \alpha_n \cdot x_n \in E_0\right),$$

and that $(x'_n)_{n=1}^{\infty} \subseteq E^*$ is a dual basis

$$\langle x_n, x'_m \rangle = \delta_n^m \quad (\forall n,\, m \in \mathbb{N}).$$

We define the operator $T : E \to E$ as follows:

$$T(x) := \sum_{n=1}^{\infty} \langle x, x'_n \rangle x_{n+1}. \tag{2.25}$$

Let u be a positive element of E. Since

$$\sum_{n=1}^{\infty} \langle u, x'_n \rangle = \left\langle u, \sum_{n=1}^{\infty} x'_n \right\rangle \le \gamma \cdot \|u\|,$$

it follows that $Tu \in E_0$. Hence, for every $x \in E$, $Tx \in E_0$, T is well-defined and positive. Let ξ and ρ be two real numbers, $\xi, \rho > 0$, such that for every $x \in E_0$,

$$x = \sum_{n=1}^{\infty} a_n x_n, \quad (a_n)_{n=1}^{\infty} \in \ell^1, \qquad \xi \cdot \sum_{n=1}^{\infty} |a_n| \le \|x\| \le \rho \cdot \sum_{n=1}^{\infty} |a_n|.$$

It follows that for every $n \in \mathbb{N}$ and $x \in E_+$,

$$\begin{aligned} \|T^n x\| &= \left\| \sum_{n=1}^{\infty} \langle x, x'_m \rangle x_{m+n} \right\| \\ &\le \rho \left\langle x, \sum_{n=1}^{\infty} x'_m \right\rangle \\ &\le \rho \cdot \|x\| \cdot \gamma\,. \end{aligned}$$

Therefore, T is power bounded. For every $x \in E_0$, $x = \sum\limits_{n=1}^{\infty} a_n x_n$, $(a_n)_{n=1}^{\infty} \in \ell^1$, $Tx = \sum\limits_{n=1}^{\infty} a_n x_{n+1}$. Therefore, in order to prove that T is not mean ergodic, it is enough to prove that the positive contraction $Q : \ell^1 \to \ell^1$,

$$Q((a_n)_{n=1}^{\infty} = (0, a_1, a_2, a_3, \ldots) \quad (\forall (a_n)_{n=1}^{\infty} \in \ell^1),$$

is not mean ergodic. We leave this as an exercise to the reader. This contradicts our condition that every power bounded operator $T \in \mathcal{L}(E)$ is mean ergodic. Thus, E does not contain a lattice isomorphic copy of ℓ^1 and, therefore, E is reflexive. □

2.3.4 Now we generalize Theorem 2.3.3, as in [27], where the assumption that a Banach lattice is σ-Dedekind complete was canceled. We need the following two simple lemmas on the geometry of Banach lattices.

Lemma 2.3.4. *Let E be a Banach lattice that fails to be σ-Dedekind complete. Then there exists an order bounded countable set $X \subseteq E$ of pairwise disjoint positive elements which has no supremum, and $\|e\| \geq 1$ for every $e \in X$.*

Proof. By the result of Veksler and Geĭler [128], there exist an element $v \in E^+$ and countable set $\{x_n\}_{n=1}^{\infty} \subseteq [0, v]$ of pairwise disjoint positive elements which has no supremum. The sequence $s_n = \sum_{k=1}^{n} x_k$ does not converge in norm. Indeed, if $s_n \to x$, then the element x is the supremum of $\{x_n\}_{n=1}^{\infty}$. Thus, the sequence $(s_n)_{n=1}^{\infty}$ is not Cauchy. Therefore, there exist $\varepsilon > 0$ and a strictly increasing sequence $(n_k)_{k=1}^{\infty}$ of natural numbers such that $\|s_{n_{k+1}} - s_{n_k}\| \geq \varepsilon$.

We put $A = \{e_k\}_{k=1}^{\infty}$, where $e_k = \varepsilon^{-1}(s_{n_{k+1}} - s_{n_k})$. The countable set A is bounded from above by $\varepsilon^{-1}v$ and consists of positive pairwise disjoint elements e_k satisfying $\|e_k\| \geq 1$. The set A has no supremum in E. Indeed, if

$$y = \sup\{e_k : k = 1, 2, \dots\}$$

exists, then we would have

$$\begin{aligned} y &= \sup\{\sup\{\varepsilon^{-1}\, x_n : n = n_k + 1, \dots, n_{k+1}\} : k = 1, 2, \dots\} \\ &= \varepsilon^{-1}\, \sup\{x_n : n = n_1 + 1, \dots\}. \end{aligned}$$

But the last supremum does not exist by the choice of the set $\{x_n\}_{n=1}^{\infty}$. □

Lemma 2.3.5. *Let E be a Banach lattice. Then, for every order bounded sequence $(e_n)_{n=1}^{\infty}$ of pairwise disjoint positive elements in E and for every real sequence $(a_n)_{n=1}^{\infty}$, $a_n \to 0$, the series $\sum_{n=1}^{\infty} a_n e_n$ converges in norm.*

Proof. Let $0 \leq e_n \leq u$ for some u and let $\{a_n\}_{n=1}^{\infty} \subseteq \mathbb{R}$, $a_n \to 0$. Then, for every $\varepsilon > 0$, there exists a number $n(\varepsilon)$ such that $|a_n| \cdot \|u\| \leq \varepsilon$ for all $n > n(\varepsilon)$. So we have

$$\begin{aligned} \left\| \sum_{n=n(\varepsilon)+1}^{n(\varepsilon)+m} a_n e_n \right\| &= \|\sup\{\,|a_n|\; e_n \;:\; n = n(\varepsilon)+1, \dots, n(\varepsilon)+m\}\| \\ &\leq \|u\| \cdot \max\{|a_n| \;:\; n = n(\varepsilon)+1, \dots, n(\varepsilon)+m\} \\ &\leq \varepsilon \end{aligned}$$

for all $m \in \mathbb{N}$. Consequently, the series $\sum_{n=1}^{\infty} a_n \cdot e_n$ converges in norm in E. □

2.3.5 Now, from Theorem 2.3.3, the affirmative answer to Sucheston's question for arbitrary Banach lattices follows due to the following result [27].

Theorem 2.3.6 (Emel'yanov). *Let E be a Banach lattice that fails to be σ-Dedekind complete. Then there exists a positive compact operator $A : E \to E$ such that the operator $T = I_E - A$ is power bounded and not mean ergodic.*

Proof. According to Lemma 2.3.4, in the Banach lattice E, there exists an order bounded family $\{e_n\}_{n=0}^{\infty}$ of positive pairwise disjoint elements without supremum and such that $\|e_n\| \geq 1$ for all n. Take an element $u \in E_+$ satisfying $\{e_n\}_{n=0}^{\infty} \subseteq [0, u]$.

For every $n = 0, 1, \ldots$, take a functional f_n defined in the one-dimensional subspace $\{\lambda e_n \,:\, \lambda \in \mathbb{R}\}$ of the Banach lattice E by $f_n(\lambda e_n) := \lambda$. We extend f_n to a functional $\overline{f_n}$ in E preserving the norm of f_n. Define the functional ξ_n in E as follows:

$$\xi_n(x) := \sup_k \overline{f_n}_+(x \wedge k e_n)$$

for $x \geq 0$ and $\xi_n(x) = \xi_n(x_+) - \xi_n(x_-)$ for an arbitrary $x \in E$. Next, put

$$\varphi_n(x) := \xi_n(x) + \frac{\|u\| - \xi_n(u)}{\xi_0(u)}\, \xi_0(x)$$

for all $x \in E$ and $n = 1, 2, \ldots$. We observe that

$$\varphi_n \geq 0, \quad \varphi_n(u) = \|u\|, \quad \varphi_n(e_n) = 1, \quad \|\varphi_n\| \;\leq\; 1 + \|u\| \tag{2.26}$$

for all natural n, and $\varphi_n(e_m) = 0$, whenever $n, m \in \mathbb{N}, n \neq m$. We explain

$$\|\varphi_n\| \leq 1 + \|u\|$$

only. Note that

$$\begin{aligned} \|\xi_n\| \leq \|\overline{f}_{n+}\| &\leq \|\overline{f}_n\| = \|f_n\| \\ &= \|e_n\|^{-1} \leq 1 \end{aligned}$$

and

$$\begin{aligned} 1 = \xi_n(e_n) &\leq \xi_n(u) \\ &\leq \|\xi_n\|\;\|u\| \\ &\leq \|u\| \end{aligned}$$

for every $n = 0, 1, \ldots$. Consequently,

$$\begin{aligned} \|\varphi_n\| &\leq \|\xi_n\| + \frac{\|u\| - \xi_n(u)}{\xi_0(u)}\|\xi_0\| \\ &\leq \|\xi_n\| + \frac{\|u\|}{\xi_0(u)}\|\xi_0\| \\ &\leq \|\xi_n\| + \|u\|\|\xi_0\| \\ &\leq 1 + \|u\| \end{aligned}$$

for all $n \in \mathbb{N}$.

Let (α_n) be an arbitrary sequence of reals satisfying $0 < \alpha_n \leq 1$ and $\alpha_n \to 0$. Define the operator $A : E \to E$ as

$$Ax := \sum_{n=1}^{\infty} \alpha_n \varphi_n(x)\ e_n \qquad (x \in E).$$

The operator A is well defined, due to Lemma 2.3.5, since $\alpha_n \varphi_n(x) \to 0$ for all $x \in E$. It is clear that A is a positive compact operator.

Next, consider the operator $T = I_E - A$. An easy computation shows that

$$T^k y = y - \sum_{n=1}^{\infty}\ [1 - (1 - \alpha_n)^k] \varphi_n(y)\ e_n \tag{2.27}$$

for all $k \in \mathbb{N}$ and $y \in E$. From (2.26) and (2.27), it ensues that

$$\begin{aligned} |T^k y| &\leq |y| + \|y\|\ (\sup_n \|\varphi_n\|) \sum_{n=1}^{\infty}\ [1 - (1 - \alpha_n)^k]\ e_n \\ &\leq |y| + \|y\|\ (1 + \|u\|) \sum_{n=1}^{\infty}\ [1 - (1 - \alpha_n)^k]\ e_n \\ &= |y| + \|y\|\ (1 + \|u\|)\ \sup_n\ [1 - (1 - \alpha_n)^k]\ e_n \\ &\leq |y| + \|y\|\ (1 + \|u\|) u\,. \end{aligned}$$

Thus,

$$\|T^k y\| \leq (1 + \|u\| + \|u\|^2) \cdot \|y\|$$

for all $k \in \mathbb{N}$ and $y \in E$. Hence, T is a power bounded operator.

We now show that the operator T is not mean ergodic. Assume the contrary. Let T be mean ergodic. Then $\mathcal{A}_n^T x \to \overline{x} \in E$ for all $x \in E$. In particular, the norm limit $v = \lim_{n\to\infty} \mathcal{A}_n^T u$ exists. From (2.27), it follows that $T^k u \downarrow$. Furthermore,

$$v = \inf_n \mathcal{A}_n^T u = \inf_n T^n u. \tag{2.28}$$

In virtue of (2.26) and (2.27), we have for every $k, n \in \mathbb{N}$,

$$v \leq T^k u \leq u - [1 - (1 - \alpha_n)^k]\ \varphi_n(u)\ e_n. \tag{2.29}$$

Passing to the limit in (2.29) as $k \to \infty$, we obtain

$$v \leq u - \varphi_n(u)\ e_n = u - \|u\|\ e_n \tag{2.30}$$

for all $n \in \mathbb{N}$. We show that $v = \inf\{u - \|u\|\ e_n : n \in \mathbb{N}\}$. Let an element $x \in E$ satisfy the condition $x \leq u - \|u\|\ e_n$ for all n. Then, by (2.26), we have

$$x \leq u - [1 - (1 - \alpha_n)^k]\ \varphi_n(u)\ e_n$$

for all $n, k \in \mathbb{N}$. From this, in accordance with (2.27), we obtain

$$\begin{aligned} x &\le \inf_n \; (u - [1-(1-\alpha_n)^k] \; \varphi_n(u) \; e_n) \\ &= u - \sup_n \; [1-(1-\alpha_n)^k] \; \varphi_n(u) \; e_n \\ &= u - \sum_{n=1}^{\infty} \; [1-(1-\alpha_n)^k] \; \varphi_n(u) \; e_n \\ &= T^k u \end{aligned}$$

for all $k \in \mathbb{N}$. Then, by (2.28),

$$x \le \inf_n T^n u = \inf_n \mathcal{A}_n^T u = v.$$

Due to (2.30) and the fact that x is an arbitrary lower bound of the set

$$\{u - \|u\| \; e_n\}_{n=1}^{\infty},$$

we obtain $v = \inf\{u - \|u\| \; e_n : n \in \mathbb{N}\}$. Consequently, there exists an element

$$\begin{aligned} \|u\| \sup\{e_n : n \in \mathbb{N}\} &= \sup\{\|u\| e_n : n \in \mathbb{N}\} \\ &= u - \inf\{u - \|u\| e_n : n \in \mathbb{N}\} \\ &= u - v \,. \end{aligned}$$

Hence the set $\{e_n\}_{n=1}^{\infty}$ has a supremum in E. So the set $\{e_n\}_{n=0}^{\infty}$ has a supremum too. This contradicts the choice of this set, which shows that the operator $T = I_E - A$ is not mean ergodic. The proof is completed. □

2.3.6 Now we discuss asymptotic properties of power bounded operators whose peripheral spectrum is $\{1\}$. From Proposition 1.1.23, it follows directly, that:

Proposition 2.3.7. *On a reflexive Banach space, every power bounded operator T with the peripheral spectrum $\sigma_\pi(T) = \{1\}$ is strongly stable.* □

We do not know any example of a non-reflexive Banach space in which every power bounded operator T with $\sigma_\pi(T) = \{1\}$ is strongly stable. According to [33], this cannot occur in Banach lattices.

Theorem 2.3.8 (Emel'yanov–Räbiger–Wolff). *Let E be a Banach lattice. Then the following conditions are equivalent*:

(i) *every power bounded operator T with peripheral spectrum $\sigma_\pi(T) = \{1\}$ in E is strongly stable*;

(ii) *E is reflexive.*

Proof. (i) ⇒ (ii): (I) Assume that the Banach lattice E is not σ-Dedekind complete. Consider the operator T constructed in the proof of Theorem 2.3.6. The operator T is power bounded and not mean ergodic (thus T is not strongly stable), and it is easy to see, that the peripheral spectrum of T is $\{1\}$. Note that the operator T is regular, but is not positive.

(II) Assume that E is σ-Dedekind complete, but the norm on E is not order continuous. Then, by the Räbiger's result [98], there exists an operator T in E satisfying $0 \leq T \leq I_E$ which is not mean ergodic. Obviously, the operator T is power bounded and not strongly stable, and $\sigma_\pi(T) = \{1\}$.

(III) So we can assume that the norm on E is order continuous. To prove reflexivity of E it is sufficient to show that neither c_0 nor ℓ^1 is lattice embeddable into E (see [85, Thm. 2.4.15]).

Assume that there exist a sublattice F of E and lattice isomorphism V from F onto c_0. Since E is countably order complete, by [85, Cor.2.4.3], F is the range of a positive projection P. Set

$$G = V^{-1} \circ T_1 \circ VP\,,$$

where T_1 is the operator constructed in Example 1.3.8. Then G is a power bounded positive operator which is not mean ergodic, and the peripheral spectrum of G is $\{1\}$.

Now suppose that ℓ^1 is lattice embeddable into E. In this case, we argue as in the proof of Theorem 2.3.3. We only have to replace the operator $Q : \ell^1 \to \ell^1$ given by

$$P((a_n)_n) = (0, a_1, a_2, a_3, \ldots)$$

by the operator

$$Q = \alpha I_E + (1 - \alpha)P$$

for some $0 < \alpha < 1$. Then the operator T given by (2.25) is well defined, power bounded, positive, and not mean ergodic. Moreover, $\sigma_\pi(T) = \{1\}$. Altogether this shows that E is reflexive.

(ii) ⇒ (i): This follows from Proposition 2.3.7. □

2.3.7 The natural question arises in connection with Theorem 2.2.5:

Does the condition (2.18) *characterize KB-spaces among Banach lattices*?

In general, the answer is unknown. Even if we know that any positive power bounded operator T in a Banach lattice E is mean ergodic, we can not conclude that E is a KB-space. However, such a characterization is possible, whenever we consider σ-Dedekind complete Banach lattices (see [34], [50], and [8]).

Theorem 2.3.9. *Let E be a σ-Dedekind complete Banach lattice. Then the following conditions are equivalent*:

a) *E is a KB-space*;

b) *Any positive power bounded operator T in E, which satisfies*

$$\lim_{n\to\infty} \operatorname{dist}(\mathcal{A}_n^T x, [-g,g] + \eta B_E) = 0 \quad (\forall x \in B_E)$$

for some $g \in E_+$ and $0 \le \eta < 1$, is mean ergodic;

c) *Any positive operator T in E, which satisfies*

$$\lim_{n\to\infty} \operatorname{dist}(T^n x, [-g,g] + \eta B_E) = 0 \quad (\forall x \in B_E)$$

for some $g \in E_+$ and $0 \le \eta < 1$, is mean ergodic.

Proof. a) $\Rightarrow$ b): It follows from Theorem 2.2.5.

b) $\Rightarrow$ c): It is obvious.

c) $\Rightarrow$ a): Assume that E is not a KB-space. If the norm on E is not order continuous, then there exists a disjoint order bounded sequence $(e_n)_{n=1}^\infty$ of E_+ which does not converge to 0 in norm [85, Thm.2.4.2.]. Without loss of generality we may assume that $\|e_n\| = 1$ and $e_n \le u$ for some $u \in E$ and all n. By [110, Exer.II.18.b], there exists a disjoint normalized sequence $(\psi_n)_{n=1}^\infty$ in E'_+ such that $\psi_n(e_m) = 0$ for $m \ne n$ and $\psi_n(e_n) \ge 1/2$. We set

$$\varphi_n = \frac{\psi_n}{\psi_n(e_n)}.$$

Then $\|\varphi_n\| \le 2$ and $\varphi_n(e_m) = \delta_{n,m}$. The map $U : \ell^\infty \to E$ given by

$$Uf = \sup_n \{f_n e_n : n \in \mathbb{N}\}$$

is a well-defined topological lattice isomorphism [85, Lemma 2.3.10(ii)]. Define $V : E \to \ell^\infty$ by

$$(Vx)_n := \varphi_n(x).$$

Then $\|V\| \le 2$ and $V \circ U = Id$ on ℓ^∞. Consider the left shift L in ℓ^∞. L is not mean ergodic and satisfies

$$\lim_{n\to\infty} \operatorname{dist}(L^n x, [-\mathbb{1}, \mathbb{1}]) = 0 \quad (\forall\, x \in B_{\ell^\infty}),$$

where $\mathbb{1}$ is the sequence in ℓ^∞ identically equals 1. Then

$$T = U \circ L \circ V$$

is a positive power bounded operator in E which is not mean ergodic and satisfies

$$\lim_{n\to\infty} \operatorname{dist}(T^n x, [-U(\mathbb{1}), U(\mathbb{1})]) = 0 \quad (\forall\, x \in B_E).$$

Thus the norm on E is order continuous. By [85, Thm.2.4.12], there exists a sublattice F of E and a lattice isomorphism V_0 from F onto c_0, and, by [85, Cor.2.4.3], F is the range of a positive projection P. Set

$$S = V_0^{-1} \circ T_\eta \circ V_0 \circ P,$$

where T_η is the operator in c_0 constructed in Example 1.3.8 and η satisfies

$$0 < \eta \|V_0^{-1}\| \|V_0\| \|P\| < 1.$$

Then S is a positive power bounded operator, and

$$\lim_{n \to \infty} \operatorname{dist}(S^n x, [-V_0^{-1} e_1, V_0^{-1} e_1] + \eta \|V_0^{-1}\| \|V_0\| \|P\| \cdot B_E) = 0 \quad (\forall\, x \in B_E).$$

The operator T_η is not mean ergodic in c_0. Hence the operator S in E is also not mean ergodic. □

Related Results and Notes

2.3.8 In the following exercise, we consider examples of positive operators in ℓ^1, ℓ^∞, and c_0, which are not mean ergodic.

Exercise 2.3.10. Let operators $Q : \ell^1 \to \ell^1$, $S : \ell^\infty \to \ell^\infty$, and $R : c_0 \to c_0$ be defined as

$$Q((a_n)_{n=1}^\infty = (0, a_1, a_2, a_3, \ldots) \quad (\forall (a_n)_{n=1}^\infty \in \ell^1);$$

$$S((a_n)_{n=1}^\infty = (a_2, a_3, \ldots) \quad (\forall (a_n)_{n=1}^\infty \in \ell^\infty);$$

$$R((a_n)_{n=1}^\infty = (a_1, a_1, a_2, a_3, \ldots) \quad (\forall (a_n)_{n=1}^\infty \in c_0).$$

Show that Q, S, and R are not mean ergodic.

By using these operators, Zaharopol [132] had proved Theorem 2.3.3. In connection with Theorem 2.3.3 and Theorem 2.3.8, the following question arises.

Open Problem 2.3.11. *Let E be a Banach lattice E. Are the following conditions equivalent*:

(i) *E is reflexive*;

(ii) *every positive power bounded operator in E is mean ergodic*;

(iii) *every power bounded operator T such that $\sigma_\pi(T) = \{1\}$ in E is strongly stable*?

2.3.9 Räbiger has characterized in [98] the order continuity of the norm on a Banach lattice E by the mean ergodicity of all operators T satisfying $0 \le T \le I_E$, whenever E possesses a topological orthogonal system $(u_\alpha)_{\alpha \in A}$. Namely, he proved the following:

Theorem 2.3.12 (Räbiger). *Let E be a Banach lattice containing a t.o.s. If the norm on E is not order continuous then there exists an operator T in E satisfying $0 \le T \le Id_E$ which is not ergodic.* □

This guarantees that the center $Z(E)$ of E, which is the linear span of

$$\{T : 0 \leq T \leq I_E\},$$

is large enough. There are, however, Banach lattices with $\dim(Z(E)) = 1$ (for an example of an AM-space with this property, see [51]). Theorem 2.3.1 characterizes the order continuity of the norm on a Banach lattice in full generality.

From Theorems 2.3.3 and 2.3.6, and the fact that every power bounded operator in a reflexive Banach space is mean ergodic, we obtain directly the following:

Theorem 2.3.13 (Emel'yanov). *For every Banach lattice E, the following conditions are equivalent*:

(i) *every power bounded operator $T : E \to E$ is mean ergodic*;

(ii) *every power bounded regular operator $T : E \to E$ is mean ergodic*;

(iii) *E is reflexive.* □

Let E be a (not necessary normed) vector lattice and let $T : E \to E$ be a linear operator. We call the operator T *mean order ergodic* if for every $x \in E$ the sequence $(\mathcal{A}_n^T x)_{n=0}^{\infty}$ order converges in E. As it is easy to see, the operator T constructed in the proof of Theorem 2.3.6 is power order bounded and not mean order ergodic. Therefore a Banach lattice in which every power order bounded operator is mean order ergodic is necessarily σ-Dedekind complete.

2.3.10 Fonf, Lin, and Wojtaszczyk [45] proved the following result, which is closely related to the Sacheston question.

Theorem 2.3.14 (Fonf–Lin–Wojtaszczyk). *Let E be a Banach space. Then X is reflexive if and only if each power bounded operator defined in a closed subspace of X is mean ergodic.* □

Also, they gave the following partial answer to Sacheston's question.

Theorem 2.3.15 (Fonf–Lin–Wojtaszczyk). *Let E be a Banach space with basis. Then X is reflexive if and only if each power bounded operator in X is mean ergodic.* □

2.3.11 Results of this section should have considerable analogues for C_0-semigroups. It is easy to formulate those analogues, and we leave this to the reader.

Chapter 3

Positive semigroups in L^1-spaces

In this chapter, we investigate asymptotic properties of one-parameter positive semigroups in $L^1(\Omega, \Sigma, \mu)$, where (Ω, Σ, μ) is a measure space with a σ-finite measure μ. In the last section, we shall also consider the theory of Markov semigroups in so-called non-commutative L^1-spaces. For one-parameter positive semigroups in L^1-spaces, there is a rich theory, which includes many results on the existence of invariant densities, criteria for asymptotic stability, decomposition theorems, etc. (cf. [71]).

The choice of results presented in this chapter is motivated mainly by the author's research interests, and it does not reflect the present state of the very broad asymptotic theory of positive semigroups in L^1-spaces. We send the reader for many other important aspects of this theory and for their applications to books of Foguel [43], Krengel [67], Lasota and Mackey [71], and Schaefer [110].

3.1 Mean ergodicity of positive semigroups in L^1-spaces

In this section, we recall some background facts about L^1-spaces and Markov operators in them. We discuss criteria for existence of an invariant density for a Markov semigroup, study interactions between the mean ergodicity and the weakly almost periodicity for positive semigroups in $L^1(\Omega, \Sigma, \mu)$, and present several results about conditions under which one-parameter positive semigroups in L^1-spaces are mean ergodic. Then we investigate conditions for mean ergodicity of Markov and Frobenius–Perron semigroups.

3.1.1 Let (Ω, Σ, μ) be a measure space equipped with a σ-finite measure μ, and let $L^1(\Omega, \Sigma, \mu)$ be the set (we call it, in short, an *L^1-space*) of all real-valued Lebesgue-integrable functions on Ω. We use the term "L^1-space" in this and the next section only for spaces $L^1(\Omega, \Sigma, \mu)$. As usual, the measurable functions which coincide almost everywhere (a.e. for short) are identified.

By L^1_+ we denote the positive cone of L^1, and by L^1_0 the set of all functions from L^1 possessing a zero integral. A function $f \in L^1_+$ is said to be *strictly positive* if $f > 0$ a.e. The term *support* of a measurable function f will be used for the set $\{f > 0\}$. Given an $A \in \Sigma$, we denote its *indicator function* (or *characteristic function*) by $1\!\!1_A$.

Positive norm 1 elements of $L^1(\Omega, \Sigma, \mu)$ are called *densities*. We denote the set of all densities on (Ω, Σ, μ) by $\mathcal{D} = \mathcal{D}(\Omega, \Sigma, \mu)$. The set $\mathcal{D}$ is obviously convex and closed. Moreover, this set is weakly compact if and only if $\dim(L^1(\Omega)) < \infty$.

We shall use a simple inequality that holds due to additivity of the norm on $L^1_+(\Omega)$. Given a non-empty $\Xi \subseteq L^1_+(\Omega)$, then

$$\inf\{\|\xi\| : \xi \in \Xi\} \le \|x\| \tag{3.1}$$

for all $x \in \overline{\mathrm{co}}(\Xi)$.

3.1.2 The following lemma (cf. [63]) will be applied below for obtaining integral versions of Theorems 3.1.14 and 3.2.4.

Lemma 3.1.1. *Let $(f_n)_{n=1}^{\infty}$ be a sequence of densities in $L^1(\Omega, \Sigma, \mu)$. Then the following conditions are equivalent*:

(i) *there exists $y \in L^1_+$ such that* $\lim\sup_{n\to\infty} \mathrm{dist}(f_n, [-y, y]) \le \eta$ *for some $\eta \in \mathbb{R}$, which satisfies $0 \le \eta < 1$;*

(ii) *there exists $y \in L^1_+$ such that* $\lim\sup_{n\to\infty} \|(f_n - y)_+\| \le \eta$ *for some $\eta \in \mathbb{R}$, which satisfies $0 \le \eta < 1$;*

(iii) *there exist $\delta > 0$, $\lambda < 1$, and $A \in \Sigma$, $\mu(A) < \infty$, such that there is an integer $n_0 = n_0(\delta, \eta, A)$, for which*

$$\int_{(\Omega - A)\cup B} f_n \, d\mu \le \lambda$$

for $n \ge n_0$ and $\mu(B) \le \delta$.

Proof. The equivalence (i) $\Leftrightarrow$ (ii) is obvious.

(iii) $\Rightarrow$ (ii): Take $\delta > 0$, $\lambda < 1$, and $A \in \Sigma$ as in condition (iii). Denote by $\mathcal{X}$ the set $\{f_n : n \ge n_0\}$. Then

$$\int_{(\Omega - A)\cup D} g \, d\mu \le \eta$$

for all $g \in \mathcal{X}$, whenever $\mu(D) \le \delta$. Suppose that (ii) does not hold. Then, for each $k \in \mathbb{N}$, there is an element $g_k \in \mathcal{X}$ such that

$$\|(g_k - k1\!\!1_A)_+\| \ge 1 - 1/k\,.$$

Henceforth,

$$\begin{aligned} 1 &\leq \limsup_{k\to\infty} \left\|(g_k - k\mathbb{1}_A)_+\right\| \\ &= \limsup_{k\to\infty} \Big[\int_{\Omega-A} g_k\,d\mu + \int_C (g_k - k\mathbb{1}_A)_+ d\mu\Big] \\ &\leq \limsup_{k\to\infty} \Big[\int_{\Omega-A} g_k\,d\mu \;+ \int_{D_k:=\{x\in\Omega: g_k(x)\geq k\}} g_k\,d\mu\Big] \\ &= \limsup_{k\to\infty} \int_{(\Omega-A)\cup D_k} g_k\,d\mu \\ &\leq \lambda \\ &< 1, \end{aligned}$$

since $\mu(D_k) \leq 1/k \leq \delta$ for large enough k. The contradiction shows that (ii) holds.

(i) $\Rightarrow$ (iii): Let (i) be satisfied with an element $y \in L^1_+$ and $\eta < 1$. Take a set $A \in \Sigma$ of finite measure and $\delta > 0$ such that

$$\int_{(\Omega-A)\cup D} y\;d\mu \leq (1-\eta)/4$$

for all $D \in \Sigma$, $\mu(D) \leq \delta$. The inequality $\|(|f| - y)_+\| \leq \eta$ holds for all

$$f \in [-y, y] + \eta B_{L^1}.$$

By (i) ,

$$\operatorname{dist}(f_n, [-y,y]) \leq \eta + (1-\eta)/4 \qquad (\forall n \geq n_0),$$

then

$$\|(f_n - y)_+\| \leq \eta + (1-\eta)/4 \qquad (\forall n \geq n_0),$$

and

$$\begin{aligned} \int_{(\Omega-A)\cup D} f_n\,d\mu &\leq \int_\Omega (f_n - y)_+ d\mu \;+ \int_{(\Omega-A)\cup D)} y\;d\mu \\ &\leq \left\|(f_n - y)_+\right\| + (1-\eta)/4 \\ &< \eta + (1-\eta)/2 \\ &= (1+\eta)/2 : \\ &= \lambda \\ &< 1 \end{aligned}$$

for all $n \geq n_0$, whenever $\mu(D) \leq \delta$. □

3.1.3 A linear operator T in an L^1-space is called a *Markov operator* if

$$T(\mathcal{D}) \subseteq \mathcal{D}.$$

A semigroup $\mathcal{T}$ in an L^1-space is called a *Markov semigroup* if it consists of Markov operators. It is clear that a linear operator T is a Markov operator if and only if it satisfies the following conditions:

$$Tf \geq 0 \quad \text{and} \quad \|Tf\| = \|f\| \quad \text{for} \quad \text{all} \quad f \in L^1_+.$$

For any Markov operator T there holds $T(L^1_0) \subseteq L^1_0$.

A density u is called *T-invariant* for a Markov operator T if $Tu = u$. Given a Markov semigroup $\mathcal{T}$, then a density u is called *$\mathcal{T}$-invariant* if $T_t u = u$ for all $T_t \in \mathcal{T}$. We denote the set of all $\mathcal{T}$-invariant densities by $\mathcal{D}_\mathcal{T}$. The notion of invariant density plays the important role in the asymptotic theory of Markov semigroups. The structure of the set $\mathcal{D}_\mathcal{T}$ can be very different. Of course, it is possible that $\mathcal{D}_\mathcal{T} = \emptyset$ or $\mathcal{D}_\mathcal{T} = \mathcal{D}$. Below we use the following simple and well-known fact.

Proposition 3.1.2. *Let $r = \dim(\mathcal{D}_\mathcal{T}) < \infty$, then there exist r extreme points $\{d_i\}_{i=1}^r$ of the set $\mathcal{D}_\mathcal{T}$ such that $\mathcal{D}_\mathcal{T} = \mathrm{co}\{d_i\}_{i=1}^r$. Moreover, the densities d_i are pairwise disjoint, i.e., $d_k \wedge d_l = 0$, whenever $k \neq l$.* □

In general, when $\dim(\mathcal{D}_\mathcal{T})$ is not necessarily finite, it is known that the set $\mathrm{Fix}(\mathcal{T})$ of all fixed vectors of $\mathcal{T}$ is a Banach sublattice in $L^1(\Omega, \Sigma, \mu)$ (see [110]). Moreover, $\mathrm{Fix}(\mathcal{T})$ is itself a space $L^1(\Omega, \Sigma_1, \mu)$, where Σ_1 is an appropriate σ-subalgebra of Σ.

Any mean ergodic Markov semigroup $\mathcal{T}$ possesses at least one invariant density, that can be obtained if we start from arbitrary density f and consider an element $u = \lim\limits_{t\to\infty} \mathcal{A}_t^\mathcal{T} f$. Obviously, $\mathcal{T}u = u$. On the other hand, we have the following well-known criterion for weak pre-compactness in L^1:

Theorem 3.1.3 (Danford–Schwartz). *A subset of L^1-space is weakly pre-compact if and only if it is almost order bounded.* □

This theorem and Theorem 1.1.7 imply directly:

Proposition 3.1.4. *Any one-parameter Markov semigroup $\mathcal{T}$ in an L^1-space possessing a strictly positive invariant density is mean ergodic.* □

3.1.4 Let $\mathcal{T}$ be a one-parameter Markov semigroup in $L^1(\Omega, \Sigma, \mu)$ such that the set $\mathcal{D}_\mathcal{T}$ of its invariant densities is non-empty. Take a finite measure μ_1, which is equivalent to the initial σ-finite measure μ on Ω, and define

$$\alpha := \sup\{\mu_1(E) : E = \{d > 0\} \text{ for some } d \in \mathcal{D}_\mathcal{T}\}.$$

Take a subset $\{d_n\}_{n=1}^\infty$ in $\mathcal{D}_\mathcal{T}$ satisfying $\mu_1(\{d_n > 0\}) \to \alpha$ and put $a = \sum\limits_{n=1}^\infty 2^{-n} d_n$. Then $a \in \mathcal{D}_\mathcal{T}$ and $\mu_1(\{a > 0\}) = \alpha$, i.e., a is a $\mathcal{T}$-invariant density of maximal support. Thus we obtain the following proposition.

Proposition 3.1.5. *Any one-parameter Markov semigroup possessing at least one invariant density possesses an invariant density of maximal support.* □

3.1.5 The following lemma gives an important property of Cesàro convergence for Markov semigroups.

Lemma 3.1.6. *Let $\mathcal{T} = (T_t)_{t\in J}$ be a one-parameter Markov semigroup in $L^1(\Omega, \Sigma, \mu)$ and let $x \in L^1_+(\Omega)$, $x \neq 0$, be such that the norm limit $y = \lim_{t\to\infty} \mathcal{A}^{\mathcal{T}}_t x$ exists. Denote by P the projection $Pf := f \cdot 1\!\!1_{\{y=0\}}$. Then*

$$\lim_{t\to\infty} \|P \circ T_t\, x\| = 0.$$

Proof. Applying the inequality (3.1) to the subset

$$\Xi := \{P \circ T_t\, x : t \in J\}$$

of $L^1_+(\Omega)$, we obtain that

$$\inf\{\|P \circ T_t\, x\| : t \in J\} \leq \inf_t \|P \circ \mathcal{A}^{\mathcal{T}}_t x\| \qquad (\forall\, \tau \in J).$$

Hence $\lim_{t\to\infty} \|P \circ \mathcal{A}^{\mathcal{T}}_t x\| = 0$ implies that $\lim_{t\to\infty} \|P \circ T_t\, x\| = 0$, which is required. □

Mean ergodic Markov semigroups have the following important property, which goes back to Helmberg [55]. The proof of it follows directly from Lemma 3.1.6.

Theorem 3.1.7 (Helmberg). *Let $\mathcal{T}$ be a one-parameter mean ergodic Markov semigroup in L^1, and let $u \in \mathcal{D}$ be a $\mathcal{T}$-invariant density of maximal support. Denote by P the projection $Pf := f \cdot 1\!\!1_{\{u=0\}}$. Then*

$$\lim_{t\to\infty} \|P \circ T_t\, f\| = 0 \qquad (\forall\, f \in L^1).$$ □

3.1.6 The problem of existence of invariant densities is one of the central problems in the theory of Markov operators. There are many results which give various conditions for existence of invariant densities. We refer for them to Krengel's book [67, Sec.3.4], and to the references therein.

In the next subsection, we give a criterion for existence of an invariant density which was obtained in [28, Thm.1]. It will be applied for proving Theorem 3.1.17 on mean ergodicity of a Frobenius–Perron semigroup.

The proof of the criterion is based on an important result of Krengel. Let us recall a definition. Given a positive contraction T in $L^1(\Omega, \Sigma, \mu)$. A *weakly wandering function* h for T is an element of $L^\infty_+(\Omega, \Sigma, \mu)$ for which there exists a strictly increasing sequence $0 = k_0 < k_1 < \cdots$ of integers with

$$\|\sum_{\nu=0}^{\infty} T^{*k_\nu} h\|_\infty < \infty.$$

The following result is due to Krengel (cf. [67, Thm. 3.4.6]).

Theorem 3.1.8 (Krengel). *If T is a positive contraction in $L^1(\Omega,\Sigma,\mu)$, there exists a decomposition of Ω into two disjoint sets C and D, uniquely determined up to the null-set by the properties*:

(i) *there exists a $p \in L^1_+(\Omega,\Sigma,\mu)$ with $Tp = p$; and $\{p > 0\} = C$;*

(ii) *there exists a weakly wandering $h \in L^\infty_+(\Omega,\Sigma,\mu)$ with $\{h > 0\} = D$.* □

3.1.7 Let $\mathcal{T} = (T_t)_{t\in J}$ be a one-parameter Markov semigroup in $L^1(\Omega,\Sigma,\mu)$. The next theorem was obtained in [28].

Theorem 3.1.9 (Emel'yanov). *The following conditions are equivalent*:

(i) *$\mathcal{T}$ has an invariant density*;

(ii) $\limsup\limits_{t\to\infty} \|f - T_t f\| < 2$ *for some density f*;

(iii) $\limsup\limits_{t\to\infty} \|d - T_t g\| < 2$ *for some pair of densities d, g.*

Proof. (i) ⇒ (ii) ⇒ (iii) is trivial and, for the proof of (iii) ⇒ (ii) it is enough to pick $f = (d+g)/2$.

(ii) ⇒ (i): Remark that from the equality $\|f - g\| = \|f\| + \|g\| - 2\|f \wedge g\|$, which holds for all $f, g \in L^1_+(\Omega)$, it follows that *(ii)* is equivalent to

$$(\exists f \in \mathcal{D})\ \liminf_{t\to\infty} \|f \wedge T_t f\| > 0. \tag{3.2}$$

Thus one may assume that condition (3.2) holds for a density f.

(I) First, we consider the discrete case $\mathcal{T} = (T^n)_{n=1}^\infty$. Without loss of generality one may assume that $f \in L^\infty(\Omega)$. Applying Theorem 3.1.8 to T, obtain a decomposition of Ω into two disjoint sets C and D that satisfy the following properties:

(*) there exists $p \in L^1_+(\Omega)$ with $Tp = p$ and $C = \{p > 0\}$, and

(**) there exists a weakly wandering $h \in L^\infty_+(\Omega)$ with $D = \{h > 0\}$.

It is enough to show that $p \neq 0$. Let $\left\| \sum\limits_{\nu=1}^{\infty} T^{*k_\nu} h \right\|_\infty < \infty$ for some strictly increasing sequence k_ν of naturals (such a sequence exists in view of weakly wandering of h). Given $\varepsilon > 0$, set $A_\varepsilon := \{h \geq \varepsilon\}$, then

$$\begin{aligned}
\sum_{\nu=1}^{\infty} \|1\!\!1_{A_\varepsilon} \cdot (f \wedge T^{k_\nu} f)\| &\leq \sum_{\nu=1}^{\infty} \|1\!\!1_{A_\varepsilon} \cdot T^{k_\nu} f\| \\
&= \int\limits_{A_\varepsilon} \sum_{\nu=1}^{\infty} T^{k_\nu} f \; d\mu \\
&\leq \varepsilon^{-1} \int\limits_{\Omega} h \cdot \sum_{\nu=1}^{\infty} T^{k_\nu} f \; d\mu \\
&= \varepsilon^{-1} \Big(\sum_{\nu=1}^{\infty} T^{*k_\nu} h \Big)(f) \\
&\leq \varepsilon^{-1} \cdot \|f\| \cdot \Big\| \sum_{\nu=1}^{\infty} T^{*k_\nu} h \Big\|_\infty \\
&< \infty
\end{aligned}$$

and consequently, $\lim\limits_{\nu\to\infty} \|1\!\!1_{A_\varepsilon} \cdot (f \wedge T^{k_\nu} f)\| = 0$ for all $\varepsilon > 0$. Now, in view of $A_\varepsilon \uparrow D$ $(\varepsilon \downarrow 0)$, we obtain $\lim\limits_{\nu\to\infty} \|1\!\!1_D \cdot (f \wedge T^{k_\nu} f)\| = 0$, and

$$\begin{aligned}
\limsup_{\nu\to\infty} \int\limits_C f \wedge T^{k_\nu} f \; d\mu &\geq \limsup_{\nu\to\infty} \int\limits_\Omega f \wedge T^{k_\nu} f \; d\mu - \lim_{\nu\to\infty} \|1\!\!1_D \cdot (f \wedge T^{k_\nu} f)\| \\
&\geq \liminf_{n\to\infty} \|f \wedge T^n f\| \\
&> 0\,.
\end{aligned}$$

In particular, the set C has a positive measure. Thus $p \neq 0$ and $w = \|p\|^{-1} p$ is an invariant density of T.

(II) Now let $\mathcal{T} = (T_t)_{t\geq 0}$ be a C_0-semigroup of Markov operators. Set $T = T_1$, then condition (3.2) implies

$$\liminf_{n\to\infty} \|f \wedge T^n f\| > 0,$$

and, from part (I) of the proof, it follows that there exists a density u_1 such that $T u_1 = u_1$. Clearly $u := \int_0^1 T_t u_1 \, dt$ is a $\mathcal{T}$-invariant density, since

$$u = \int\limits_0^1 T_t u_1 \, dt = \int\limits_s^{1+s} T_t u_1 \, dt = T_s u \qquad (\forall s \geq 0). \qquad \square$$

3.1.8 Let us recall the following simple proposition which is a direct consequence of Proposition 1.1.3 and Theorem 1.1.13.

Proposition 3.1.10. *Let $\mathcal{T} = (T_t)_{t\in J}$ be a one-parameter bounded positive semigroup in an L^1-space, such that an operator T_ξ is mean ergodic or (weakly) almost periodic for some $\xi > 0$. Then the semigroup $\mathcal{T}$ possesses the same property.* □

By Proposition 1.1.19, every weakly almost periodic operator and every weakly almost periodic C_0-semigroup is mean ergodic. The converse is not true in general. However, it is still true in the case of bounded positive semigroups in $L^1(\Omega)$. This result was obtained by Komornik [63, Prop. 1.4(i)] and independently by Kornfeld, Lin [65, Thm. 1.2]. The proof below follows the paper [28].

Theorem 3.1.11 (Komornik–Kornfeld–Lin). *Let $\mathcal{T} = (T_t)_{t\in J}$ be a one-parameter positive bounded mean ergodic semigroup in $L^1(\Omega, \Sigma, \mu)$. Then $\mathcal{T}$ is weakly almost periodic.*

Proof. It is enough to show that the orbit $\{T_t f\}_{t\in J}$ is conditionally weakly compact for each $f \in L^1_+(\Omega)$.

Fix $f \in L^1_+(\Omega)$ and let $u = \lim\limits_{t\to\infty} \mathcal{A}^{\mathcal{T}}_t f$. If $u = 0$, set $Q = I := Id|_{L^1(\Omega)}$. If $u \neq 0$, take the u-support projection $P = P_u$:

$$P_u g = \mathbb{I}_{\{u>0\}} \cdot g \qquad (\forall g \in L^1(\Omega)).$$

It is clear that P satisfies $0 \le P \le I$, and $T_t \circ P = P \circ T_t \circ P$ since $L^1(\{u > 0\})$ is $\mathcal{T}$-invariant. Set $Q = I - P$ and notice that $Q \circ T_t = Q \circ T_t \circ Q$ for all $t \in J$.

Since $\lim\limits_{t\to\infty} \|u - \mathcal{A}^{\mathcal{T}}_t f\| = 0$ and $Qu = 0$, we have

$$\lim_{t\to\infty} \|Q \circ \mathcal{A}^{\mathcal{T}}_t f\| = 0\,.$$

Applying the inequality (3.1) to the subset $\Xi := \{Q \circ T_t\, f\}_{t\in J}$, we obtain that

$$\|Q \circ T_{n_i} f\| \to 0 \qquad (i \to \infty)$$

for some increasing sequence $(n_i)_{i=1}^{\infty}$ and, consequently,

$$\begin{aligned}
\limsup_{t\to\infty} \|Q \circ T_t\, f\| &= \limsup_{t\to\infty} \|Q \circ T_t \circ T_{n_i} f\| \\
&= \limsup_{t\to\infty} \|Q \circ T_t \circ Q \circ T_{n_i} f\| \\
&\le \sup_{t\in J} \|T_t\| \cdot \|Q \circ T_{n_i} f\| \\
&\to 0 \qquad (i \to \infty).
\end{aligned}$$

Thus $\lim\limits_{t\to\infty} \|Q \circ T_t\, f\| = 0$. In the case $u = 0$, the proof is finished already, since $Q = I$. Let $u \neq 0$. The set $\bigcup\limits_{l=1}^{\infty} [-lu, lu]$ is norm-dense in $L^1(\{u > 0\}) = P(L^1(\Omega))$. By invariance of u,

$$T_t([-lu, lu]) \subseteq [-lu, lu] \qquad (\forall t \in J),$$

and since $\mathcal{T}$ is bounded, for each $\varepsilon > 0$ there exists l_ε such that

$$\limsup_{t\to\infty} \operatorname{dist}(T_t\, f, [-lu, lu]) = \limsup_{t\to\infty} \operatorname{dist}(P \circ T_t\, f, [-lu, lu]) \le \varepsilon$$

for all $l \ge l_\varepsilon$. Hence $\{T_t f\}_{t\in J}$ is conditionally weakly compact since $[-lu, lu]$ is weakly compact and $\varepsilon > 0$ was chosen arbitrarily. □

The following result is due to Derriennic and Krengel [24] and it follows directly from Proposition 1.1.3 and Theorem 3.1.11. Remark that this corollary can be obtained independently from Theorem 2.1.14.

Corollary 3.1.12 (Derriennic–Krengel). *Let T be a power bounded positive operator in an L^1-space. Given $m \in \mathbb{N}$, then T is mean ergodic if and only if T^m is mean ergodic.* □

3.1.9 It is important in many cases to find conditions under which a positive operator T is mean ergodic and the space $\operatorname{Fix}(T)$ of all T-fixed vectors has finite dimension. For this aim, the following theorem is useful. Indeed, this theorem is a partial case of Theorem 2.2.5, if we additionally assume that the semigroup $\mathcal{T}$ is bounded (since every L^1-space is a KB-space). Here we prove it directly.

Theorem 3.1.13 (Emel'yanov–Wolff). *Let $\mathcal{T}$ be a one-parameter positive semigroup in L^1-space such that there exist an element $y \in L^1_+$ and real η with $0 \le \eta < 1$ satisfying*

$$\limsup_{t\to\infty} \left\| (\mathcal{A}_t^{\mathcal{T}} f - y)_+ \right\| \le \eta$$

for every density f. Then the semigroup $\mathcal{T}$ is mean ergodic.

Proof. We prove this theorem only in the case when $\mathcal{T} = (T^n)_{n=1}^\infty$. The continuous case can be easily reduced to the discrete case (we leave this as an exercise to the reader).

Denote our L^1-space by E. Due to Theorem 1.1.11, it is enough to check that, for every T^*-fixed point $0 \ne \psi \in E^* = L^\infty(\Omega, \Sigma, \mu)$, there exists a T-fixed point $w \in E$ which satisfies $\langle \psi, w\rangle \ne 0$.

Let $E^* \ni \psi \ne 0$, $T^*\psi = \psi$. We may assume that $\|\psi_+\| = \|\psi\| = 1$. Set $\varepsilon := (1-\eta)/3$ and take some $f \in E$ which satisfies $\|f\| = 1$ and $\langle \psi_+, f\rangle \ge 1 - \varepsilon$. We have $\||f|\| = \|f\| = 1$ and

$$1 \ge \langle |\psi|, |f|\rangle \ge \langle \psi_+, |f|\rangle \ge \langle \psi_+, f\rangle \ge 1 - \varepsilon.$$

Consequently,

$$\begin{aligned} \langle \psi, |f|\rangle &= \langle 2\psi_+, |f|\rangle - \langle |\psi|, |f|\rangle \\ &\ge 2(1-\varepsilon) - 1 \\ &= 1 - 2\varepsilon\,. \end{aligned}$$

Let $f'' \in E^{**}$ be a w^*-cluster point of $(\mathcal{A}_n^T|f|)_{n=1}^\infty$. Then f'' obviously satisfies $T^{**}f'' = f''$. Since

$$\limsup_{n\to\infty} \operatorname{dist}(\mathcal{A}_n^T|f|, [0,y]) \le \eta$$

and $[0,y]$ is weakly compact in E, we obtain

$$f'' \in [0,y] + \eta B_{E^{**}} \subseteq E + \eta B_{E^{**}},$$

where $B_{E^{**}}$ denotes as usual the unit ball of E^{**}. Take the canonical projection $P : E^{**} \to E$. Then $(I-P)f'' \in \eta B_{E^{**}}$, and

$$\begin{aligned}
\langle \psi, Pf''\rangle &= \langle \psi_+, Pf''\rangle - \langle \psi_-, Pf''\rangle \\
&= \langle f'', \psi_+\rangle - \langle (I-P)f'', \psi_+\rangle - \langle \psi_-, Pf''\rangle \\
&\ge \langle f'', \psi\rangle - \eta \\
&= \langle \psi, |f|\rangle - \eta \\
&\ge 1 - 2\varepsilon - \eta \\
&= \varepsilon > 0\,.
\end{aligned}$$

Moreover,

$$\begin{aligned}
T \circ P\, f'' &= T \circ P \circ T^{**}\, f'' \\
&\ge T \circ P \circ T^{**} \circ P\, f'' \\
&= T \circ P \circ T \circ P\, f'' \\
&= T^2 \circ P\, f'' \\
&\ge 0\,.
\end{aligned}$$

Thus the sequence $(T^m \circ P\, f'')_m$ is decreasing in $E = L^1(\Omega, \Sigma, \mu)$, and henceforth,

$$w := \lim_{m\to\infty} T^m \circ P\, f''$$

exists. Clearly $Tw = w$ and

$$\langle \psi, w\rangle = \langle \psi, Pf''\rangle > 0\,.$$

Thus T is mean ergodic. □

Remark that this proof uses the additivity of the norm in an L^1-space, and it cannot be generalized to KB-spaces.

3.1.10 The result of Theorem 3.1.13 can be specified for Markov operators in the following form.

Theorem 3.1.14. *Let T be a Markov operator in $L^1(\Omega, \Sigma, \mu)$. Then the following assertions are equivalent*:

(i) *there exist an element $y \in L^1_+$ and real η, $0 \le \eta < 1$, such that*

$$\limsup_{n\to\infty} \left\|(\mathcal{A}_n^T f - y)_+\right\| \le \eta$$

for every density f;

(ii) *there are an integer m, decomposition $\Omega = \bigcup_{k=0}^{m} \Omega_k$ into pairwise disjoint subsets $\Omega_k \in \Sigma$, and two sequences of nonnegative functions $u_k = Tu_k \in \mathcal{D}(\Omega_k)$ and $\phi_k \in L^\infty(\Omega_0)$, $k = 1, \dots, m$, such that $\sum_{k=1}^{m} \phi_k = \mathbb{1}_{\Omega_0}$ and for every $g \in L^1(\Omega)$ the norm limit $\lim_{n\to\infty} \mathcal{A}_n^T g$ exists and can be written in the form*

$$\lim_{n\to\infty} \mathcal{A}_n^T g = \sum_{k=1}^{m} \Big[\int_\Omega (\phi_k + \mathbb{1}_{\Omega_k}) g \, d\mu \Big] u_k,$$

and, moreover, $\lim_{n\to\infty} \int_{\Omega_0} T^n g \, d\mu = 0$.

Proof. (i) $\Rightarrow$ (ii): The operator T is mean ergodic by Theorem 3.1.13. The space $\mathrm{Fix}(T)$ of all T-fixed vectors is an L^1-space as the range of a Markov projection (cf. the remark after Proposition 3.1.2). Since

$$\|(z - y)_+\| \le \eta < 1 \qquad (\forall z \in \mathcal{D} \cap \mathrm{Fix}(T)),$$

which implies

$$\|y\| \ge \|z \wedge y\| \ge 1 - \eta > 0 \qquad (\forall z \in \mathcal{D} \cap \mathrm{Fix}(T)),$$

we obtain $\dim \mathrm{Fix}(T) < \infty$. Take a maximal pairwise disjoint family $\{u_k\}_{k=1}^m$ of T-invariant densities (it is clear that $m = \dim \mathrm{Fix}(T)$), and put

$$\Omega_k := \{x \in \Omega : u_k(x) > 0\}, \qquad \Omega_0 := \Omega - \bigcup_{k=0}^{m} \Omega_k.$$

Let P be the strong limit of $(\mathcal{A}_n^T)_{n=1}^\infty$, then P may be written in the form

$$Pg = \sum_{k=1}^{m} \lambda_k(g) u_k \qquad (\forall g \in L^1),$$

where λ_k are positive linear functionals in L^1 which one may consider as elements of L^∞. Since $Pu_k = u_k$, we obtain

$$u_k = \Big[\int_\Omega \lambda_k \cdot u_k d\mu \Big] u_k.$$

Hence $\lambda_k \cdot 1\!\!1_{\Omega_k} = 1\!\!1_{\Omega_k}$. Put $\phi_k = \lambda_k - 1\!\!1_{\Omega_k}$, then the assertion (ii) is true for m, Ω_k, u_k, and ϕ_k.

Given $g \in L^1_+$, the sequence $\Big(\|1\!\!1_{\Omega_0} \cdot T^n g\|\Big)_{n=1}^{\infty}$ is obviously decreasing and satisfies $\lim_{n\to\infty} \|1\!\!1_{\Omega_0} \cdot \mathcal{A}_n^T g\| = 0$. Hence the inequality

$$\inf_n \|1\!\!1_{\Omega_0} \cdot T^n g\| \le \frac{1}{n}\sum_{k=0}^{n-1} \|1\!\!1_{\Omega_0} \cdot T^k g\| = \|1\!\!1_{\Omega_0} \cdot \mathcal{A}_n^T g\|$$

implies $\lim_{n\to\infty} \int_{\Omega_0} T^n g \, d\mu = 0$.

(ii) $\Rightarrow$ (i): It is an easy exercise. □

There is an integral version of Theorem 3.1.14, namely:

Theorem 3.1.15. *Under the same conditions as in Theorem* 3.1.14*, the assertions* (i) *and* (ii) *are equivalent to*:

(iii) *there exist $\delta > 0$, $\eta < 1$, and $A \in \Sigma$, $\mu(A) < \infty$, such that for every density f there is an integer $n(f)$ for which*

$$\int\limits_{(\Omega - A)\cup B} \mathcal{A}_n^T f \, d\mu \le \eta$$

for $n \ge n(f)$ and $\mu(B) \le \delta$.

Proof. It is enough to apply Lemma 3.1.1 to the sequence $(\mathcal{A}_n^T f)_{n=1}^{\infty}$. □

3.1.11 We give a variant of Theorem 3.1.14 for a strongly continuous Markov semigroup. The proof of it is similar to the proof of Theorem 3.1.14, and we leave it to the reader. Notice that Theorem 3.1.16 can be also easily reduced to Theorem 3.1.14.

Theorem 3.1.16. *Let $\mathcal{T} = (T_t)_{t\ge 0}$ be a C_0-semigroup of Markov operators in $L^1(\Omega, \Sigma, \mu)$. Then the following assertions are equivalent*:

(i) *there exist $y \in L^1_+$ and $\eta < 1$ such that*

$$\limsup_{t\to\infty} \left\| \left(\frac{1}{t} \int\limits_0^t T_\tau f \, d\tau - y \right)_+ \right\| \le \eta$$

for every density f;

(ii) *there are an integer m, decomposition $\Omega = \bigcup\limits_{k=0}^{m} \Omega_k$ onto pairwise disjoint subsets $\Omega_k \in \Sigma$, and two sequences of nonnegative functions $u_k = \mathcal{T} u_k \in$*

$\mathcal{D}(\Omega_k)$, $\phi_k \in L^\infty(\Omega_0)$, $k = 1, \ldots, m$, such that $\sum\limits_{k=1}^{m} \phi_k = 1\!\!1_{\Omega_0}$, and for every $g \in L^1(\Omega)$ the norm limit $\lim\limits_{t\to\infty} \frac{1}{t}\int_0^t T_\tau g d\tau$ exists and can be written in the form

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t T_\tau g\, d\tau = \sum_{k=1}^{m} \Big[\int_\Omega (\phi_k + 1\!\!1_{\Omega_k}) g\, d\mu \Big] u_k,$$

moreover, $\lim\limits_{t\to\infty} \int_{\Omega_0} T_t g d\mu = 0$;

(iii) *there exist $\delta > 0$, $\eta < 1$, and $A \in \Sigma$, $\mu(A) < \infty$, such that for every density f there is $r(f) \geq 0$ for which*

$$\int_{(\Omega - A)\cup B} \Big[\frac{1}{r}\int_0^r T_\tau f\, d\tau \Big]\, d\mu \leq \eta$$

for $r \geq r(f)$ and $\mu(B) \leq \delta$. □

3.1.12 One of the most important classes of Markov operators is the Frobenius–Perron operators. Recall that a transformation $S : \Omega \to \Omega$ is called *measurable* if $S^{-1}(A) \in \Sigma$ for all $A \in \Sigma$. A measurable transformation $S : \Omega \to \Omega$ is called *non-singular* if $\mu(S^{-1}(A)) = 0$ for all $A \in \Sigma$ such that $\mu(A) = 0$. It follows from the Radon–Nikodym theorem that for any non-singular transformation S the equality

$$\int_A P f d\mu = \int_{S^{-1}(A)} f d\mu \qquad (A \in \Sigma)$$

defines the unique operator $P : L^1(\Omega) \to L^1(\Omega)$. The operator P is called the *Frobenius–Perron operator* corresponding to S. It is easy to see that any Frobenius–Perron operator is a Markov operator.

When a semigroup $(S_t)_{t\in J}$ of non-singular transformations on (Ω, Σ, μ) is given, then $(P_{S_t})_{t\in J}$ is called the *Frobenius–Perron semigroup.*

We apply Theorem 3.1.9 for obtaining the following criterion [28, Thm.3] of mean ergodicity of a Frobenius–Perron semigroup. Let $\mathcal{P} = (P_{\tau_t})_{t\in J}$ be a discrete or strongly continuous Frobenius–Perron semigroup associated with a semigroup $(\tau_t)_{t\in J}$ of non-singular transformations $\tau_t : \Omega \to \Omega$.

Theorem 3.1.17 (Emel'yanov). *For a one-parameter Frobenius–Perron semigroup $\mathcal{P}$, the following conditions are equivalent*:

(i) *$\mathcal{P}$ is weakly almost periodic*;

(ii) *$\mathcal{P}$ is mean ergodic*;

(iii) *there exists a density* w *such that* $\limsup_{t\to\infty} \|P_{\tau_t} f - w\| < 2$ *for every density* f.

Proof. The equivalence (i) ⇔ (ii) follows from Theorem 3.1.11.

(ii) ⇒ (iii): Take a density d such that $d(x) > 0$ a.e. on Ω, then the density $w = \lim_{t\to\infty} \mathcal{A}_t^{\mathcal{P}} d$ satisfies (iii). Indeed, let $f \in \mathcal{D}$, then

$$\begin{aligned} \inf_t \|\mathcal{A}_t^{\mathcal{P}} f - w\| &\leq \lim_{t\to\infty} \|\mathcal{A}_t^{\mathcal{P}}(f - d)\| \\ &\leq \|f - d\| \\ &< 2\,. \end{aligned}$$

Since $\mathcal{A}_t^{\mathcal{P}} f \in \overline{\mathrm{co}}\{P_{\tau_t} f : t \in J\}$, the inequality above shows that there exists an element $a \in \mathrm{co}\{P_{\tau_t} f : t \in J\}$ with $\|a - w\| < 2$, and hence $\|P_{\tau_{t_0}} - w\| < 2$ for some $t_0 \in J$. But then

$$\begin{aligned} \limsup_{t\to\infty} \|P_{\tau_t} f - w\| &= \limsup_{s\to\infty} \|P_{\tau_s}(P_{\tau_{t_0}} f - w)\| \\ &\leq \|P_{\tau_{t_0}} - w\| \\ &< 2 \end{aligned}$$

which is required.

(iii) ⇒ (ii): The set $\mathcal{D}_{\mathcal{P}}$ of all $\mathcal{P}$-invariant densities is not empty by Theorem 3.1.9. Consequently, by Proposition 3.1.5, there is a $\mathcal{P}$-invariant density with maximal support, say a. Denote its support by $A = \{a > 0\}$. Let

$$A_1 = \bigcup_{t\in J} \tau_t^{-1}(A)\,.$$

Obviously, $\tau_t(A_1) \subseteq A_1$ for any $t \in J$. But

$$B := \Omega - A_1 = \{x : \tau_t x \notin A \ \forall t \in J\}$$

is also obviously invariant. Hence $L^1(B)$ is invariant for all P_{τ_t}. By *(ii)*, $1\!\!1_B \neq 0$, so Theorem 3.1.9 yields a density supported in B invariant for all P_{τ_t}, which contradicts maximality of A. Hence $\Omega = \bigcup_{t\in J} \tau_t^{-1}(A)$.

Since, obviously, $\tau_t(A) \subseteq A$ for all $t \in J$, we obtain

$$\lim_{t\to\infty} \int_{\Omega - A} P_{\tau_t} f \, d\mu = \lim_{t\to\infty} \int_{\Omega - \tau_t^{-1}(A)} f \, d\mu = 0 \qquad (\forall f \in \mathcal{D}). \tag{3.3}$$

On the other hand, the restriction $\mathcal{P}|_{L^1(A)}$ of $\mathcal{P}$ in $L^1(A)$ is mean ergodic since the semigroup $\mathcal{P}|_{L^1(A)}$ has the almost everywhere positive (on the set A) invariant density a. Consequently, (3.3) implies that $\mathcal{P}$ is mean ergodic. □

Related Results and Notes

3.1.13 Not only are Markov operators a subject of intensive study in operator theory in L^1-spaces. Positive (not necessarily linear) operators which map the positive cone $L^1_+(\Omega)$ of $L^1(\Omega)$ into itself, and semigroups consisting of such operators (*positive semigroups*) are also very popular and important. There is a rich and illuminating theory of positive contractive operators in L^1-spaces (see, for instance, [67] and [110]).

However, let us say that the asymptotic theory of linear contractive positive semigroups is very similar with the asymptotic theory of Markov semigroups. Indeed, let $\mathcal{T}$ be a positive contractive semigroup in $X = L^1(\Omega, \Sigma, \mu)$. The application of Proposition 1.1.17 to $Y = \mathbb{C}$ and $R = \mathbb{1}_\Omega \in X^* = L^\infty$ gives the Markov semigroup in the L^1-space $X \times \mathbb{C}$ with the norm $\|x \times \lambda\| := \|x\| + \|\lambda\|$. Then Proposition 1.1.18 shows that from the asymptotic point of view the consideration of positive contractive semigroups in an L^1-space does not lead to essential generalizations.

3.1.14 According to Theorem 3.1.11, any mean ergodic Markov operator in an L^1-space is weakly almost periodic. In general, this result is not true even for contractive positive operators in Banach lattices. To show this, we can take the operator T on $C(K)$ constructed in [119]. This operator is mean ergodic, but it is not weakly almost periodic, since T^2 is not mean ergodic.

Open Problem 3.1.18. *Let T be a mean ergodic positive power bounded operator on a Banach lattice with order continuous norm. Is T weakly almost periodic*?

Even for the Banach lattice c_0, the answer seems to be unknown. In the case when the answer for c_0 is negative, the following question arises.

Open Problem 3.1.19. *Let T be a mean ergodic positive power bounded operator on a KB-space. Is T weakly almost periodic*? *Does this property of positive power bounded operators characterize KB-spaces among* (σ*-Dedekind complete*) *Banach lattices*?

3.1.15 It was shown by Socala [121] that existence of invariant density for a Markov operator T in $L^1(\Omega, \Sigma, \mu)$ is equivalent to the following integral condition:

There exist $\delta > 0$ and $A \in \Sigma$, $\mu(A) < \infty$, such that

$$\limsup_{n\to\infty} \int_{(\Omega - A)\cup B} T^n f \, d\mu > 0 \tag{3.4}$$

for some density f and any $B \in \Sigma$, $\mu(B) \le \delta$.

The condition (3.4) can be considered as the integral form of the condition (ii) of Theorem 3.1.9. This theorem has the following generalization for abelian semigroups of Markov operators. The proof of it is left as an exercise for the reader.

Theorem 3.1.20. *Let $\mathcal{T}$ be an abelian semigroup of Markov operators. Then the following conditions are equivalent:*

(i) *$\mathcal{T}$ has an invariant density;*

(ii) $\limsup_{t\to\infty} \|f - T_t f\| < 2$ *for some density f;*

(iii) $\limsup_{t\to\infty} \|d - T_t g\| < 2$ *for some pair of densities d, g.* □

The following question motivated by Theorem 3.1.9 is open.

Open Problem 3.1.21. *Let $\mathcal{T}$ be a one-parameter Markov semigroup in an L^1-space for which there exists a density f such that*

$$\limsup_{t\to\infty} \|f - \mathcal{A}_t^{\mathcal{T}} f\| < 2.$$

Does $\mathcal{T}$ have an invariant density?

3.1.16 We point out that Theorems 3.1.14 and 3.1.16 hold also for an arbitrary positive semigroup in $L^1(\Omega, \Sigma, \mu)$, and that it is enough to consider only a dense subset of $\mathcal{D}$ instead of $\mathcal{D}$.

One of the first asymptotic conditions for existence of an invariant density is due to Calderon [18] and, independently, to Dowker [25]. We formulate their result in both discrete and continuous parameter cases and apply Theorem 3.1.9 in its proof.

Theorem 3.1.22 (Calderon–Dowker). *Let (Ω, Σ, μ) be a probability space. Let $(\tau_t)_{t\in J}$ be a one-parameter semigroup of non-singular transformations $\tau_t : \Omega \to \Omega$. Then the following conditions are equivalent:*

(i) *there exists an equivalent finite invariant measure;*

(ii) *for every $A \in \Sigma$, $\mu(A) > 0$ implies* $\liminf_{t\to\infty} \mu(\tau^{-t}A) > 0$.

Proof. (i) ⇒ (ii) is trivial.

(ii) ⇒ (i): Recall that existence of an equivalent finite invariant measure for $(\tau_t)_{t\in J}$ is equivalent to existence of an almost everywhere positive invariant density for the corresponding Frobenius–Perron semigroup $\mathcal{P} = (P_{\tau_t})_{t\in J}$.

Take a density $f := 1\!\!1_\Omega$. Then

$$\begin{aligned} \liminf_{t\to\infty} \int_A P_{\tau_t} f \, d\mu &= \liminf_{t\to\infty} \int_{\tau^{-t}A} f \, d\mu \\ &= \liminf_{t\to\infty} \mu(\tau^{-t}A) \\ &> 0 \end{aligned}$$

for any $A \in \Sigma$, $\mu(A) > 0$. This is possible only if

$$\liminf_{t\to\infty} \int_A (f \wedge P_{\tau_t} f) \, d\mu > 0,$$

which, due to Theorem 3.1.9, provides existence of a $\mathcal{P}$-invariant density, say u, and according to Proposition 3.1.5, one may assume that the density u is of maximal support, say $B = \{u > 0\}$. The set

$$U = \bigcup_{t \in J} \tau^{-t} B$$

is obviously τ-invariant, which means $\tau^{-t}U \subseteq U$ for all $t \in J$. Then the set $\Omega \setminus U$ is τ-invariant as well.

If $\mu(\Omega \setminus U) > 0$, then the same argument as above applying to the restriction of the semigroup $\mathcal{P}$ in $L^1(\Omega \setminus U)$ gives a density supporting on $\Omega \setminus U$, which is impossible. Consequently, $\mu(U) = 1$ and, therefore,

$$\lim_{t \to \infty} \mu\tau^{-t}(\Omega \setminus B) = 0\,.$$

Applying condition (ii) again, show that $\mu(\Omega \backslash B) = 0$, and the proof is finished. $\square$

We mention also the following (in some sense dual) characterization of existence of an invariant density, which is due to Straube [125]. It seems to be that it is also possible to prove it by using Theorem 3.1.9.

Theorem 3.1.23 (Straube). *Let (Ω, Σ, μ) be a probability space, τ a non-singular transformation $\tau : \Omega \to \Omega$. Then the following conditions are equivalent*:

(i) *there exists an equivalent probabilistic invariant μ-continuous measure on Ω*;

(ii) *there exist $\delta > 0$, and α, $0 < \alpha < 1$, such that*

$$\mu(A) < \delta \Rightarrow \sup_{k \geq 0} \mu(\tau^{-k}(A)) < \alpha \quad (\forall A \in \Sigma)\,. \qquad \square$$

3.1.17 Some interesting results on Frobenius–Perron operators are contained in [125]. The following natural question is open.

Open Problem 3.1.24. *Let $\mathcal{P}$ be a one-parameter Frobenius–Perron semigroup such that there exists a density u satisfying*

$$\limsup_{t \to \infty} \|u - \mathcal{A}_t^{\mathcal{P}} f\| < 2 \qquad (\forall f \in \mathcal{D}).$$

Is $\mathcal{P}$ mean ergodic?

The integral version of Theorem 3.1.17 can be obtained in the obvious way. Moreover, Theorem 3.1.17 has the following generalization for abelian Frobenius–Perron semigroups. We leave the proof of it as an exercise for the reader.

Theorem 3.1.25. *Let $\mathcal{P} = (P_{\tau_t})_{t \in J}$ be an abelian Frobenius–Perron semigroup. Then the following conditions are equivalent*:

(i) *$\mathcal{P}$ is weakly almost periodic*;

(ii) *$\mathcal{P}$ is mean ergodic;*

(iii) *there exists a density w such that* $\limsup_{t\to\infty} \|P_{\mathcal{T}_t} f - w\| < 2$ *for every density f.*

□

For other results about transformations of measure spaces and Frobenius–Perron operators we refer to Lasota and Mackey [71].

3.1.18 Theorems 3.1.11 and 3.1.13 can be also formulated for a representation of an abelian semigroup in the following way.

Theorem 3.1.26. *Let $\mathcal{T} = (T_t)_{t\in\mathbb{P}}$ be a bounded mean ergodic positive representation of an abelian semigroup $\mathbb{P}$ in $\mathcal{L}(L^1)$. Then $\mathcal{T}$ is weakly almost periodic.* □

Theorem 3.1.27. *Let $\mathcal{T} = (T_t)_{t\in\mathbb{P}}$ be a bounded positive representation of an abelian semigroup $\mathbb{P}$ in $\mathcal{L}(L^1)$ such that there exist an element $y \in L^1_+$ and real η, $0 < \eta < 1$, satisfying*

$$\limsup_{t\to\infty} \left\| (\mathcal{A}_t(\mathcal{T})f - y)_+ \right\| \le \eta.$$

Then the semigroup $\mathcal{T}$ is mean ergodic. □

3.1.19 Remark that the implications (i) $\Leftrightarrow$ (ii) $\Rightarrow$ (iii) of Theorem 3.1.17 are true for any Markov semigroup. The following example of Komornik [63, Ex.4.1] shows that the implication (iii) $\Rightarrow$ (ii) does not hold in general.

Example 3.1.28. Let $\Omega = \mathbb{N}$, let Σ be the algebra of all subsets of $\mathbb{N}$, and let μ be the counting measure. Thus $L_1(\Omega) = \ell^1$. Define an operator T in ℓ^1 as follows:

$$Te_{k+1} := 2^{-k} \cdot e_1 + (1 - 2^{-k}) \cdot e_{k+2} \quad (\forall k \ge 0).$$

So the defined Markov semigroup $\mathcal{T} = (T^n)_{n=1}^{\infty}$ obviously satisfies the condition (iii) of Theorem 3.1.17 with $w = e_1$. But it is easy to see that for any density d the sequence $(\mathcal{A}_n^T d)_{n=1}^{\infty}$ converges if and only if d is equal to e_1. In particular, T is not mean ergodic.

Exercise 3.1.29. Prove the assertion above.

3.1.20 We finish this section with the following problem. It is easy to see that any power bounded positive operator T on an L^1-space X such that 0 belongs to the weak-closure of $\{T^n x\}_{n=1}^{\infty}$ for all $x \in X$ satisfies a formally stronger condition:

$$\mathrm{w}-\lim_{n\to\infty} T^n x = 0 \quad (\forall x \in X).$$

In general, this is not true for power bounded operators in Banach spaces. However, it is an interesting question to extend this property on positive operators in Banach lattices, namely:

Open Problem 3.1.30. *Let T be a positive power bounded operator in a Banach lattice which satisfies $0 \in \text{w-cl}\{T^n x\}_{n=1}^{\infty}$ for each $x \in X$. Does $\text{w} - \lim_{n\to\infty} T^n x = 0$ hold for all $x \in X$?*

The similar question is for positive bounded semigroups.

3.2 Stability and lower-bounds for positive semigroups in L^1-spaces

In this section, we give several theorems about asymptotic stability in terms of lower bounds. The first of them being well known as Lasota's lower-bound criterion [69] of asymptotic stability, the others are taken from [39]. Then we discuss a theorem of Komornik and Lasota [64], which is known as the spectral decomposition theorem for Markov semigroups. This theorem plays an important role in the investigation of asymptotic behavior of many classes of Markov operators.

3.2.1 A one-parameter Markov semigroup $\mathcal{T} = (T_t)_{t\in J}$ in L^1 is called *asymptotically stable*, whenever there exists an (always $\mathcal{T}$-invariant) density u such that

$$\lim_{t\to\infty} \|T_t f - u\| = 0 \tag{3.5}$$

for every density $f \in L^1$. A function $h \in L^1_+(\Omega, \Sigma, \mu)$ is called a *lower-bound function* for a Markov semigroup $\mathcal{T}$ if

$$\lim_{t\to\infty} \|(h - T_t f)_+\| = 0$$

for every density $f \in L^1$. We say that a lower-bound function h is *non-trivial* if $h \neq 0$. Now we give a proof of the following theorem of Lasota [71, Thm. 5.6.2 and Thm. 7.4.1].

Theorem 3.2.1 (Lasota). *Let $\mathcal{T} = (T_t)_{t\in J}$ be a (not necessarily continuous if $J = \mathbb{R}_+$) one-parameter Markov semigroup in $L^1 = L^1(\Omega, \Sigma, \mu)$. Then the following assertions are equivalent:*

(i) *$\mathcal{T}$ is asymptotically stable;*

(ii) *there is $0 \neq h \in L^1_+$ such that, for any density $f \in L^1$ and for any $t \in J$, there exists $f_t \in L^1_+$ with $\lim_{t\to\infty} \|f_t\| = 0$ and $T_t f + f_t \geq h$ for all $t \in J$;*

(iii) *there exists a nontrivial lower-bound function for $\mathcal{T}$.*

Proof. (i) $\Rightarrow$ (iii): Let a density $u \in L^1$ satisfy $\lim_{t\to\infty} \|T_t f - u\| = 0$ for any density f, then u is a non-trivial lower-bound function for $\mathcal{T}$.

(iii) $\Leftrightarrow$ (ii): Let $0 \neq h \in L^1_+$ be a nontrivial lower-bound function for $\mathcal{T}$. Then for any density f, the condition *(ii)* is satisfied with $f_t := (T_t f - h)_-$ for all $t \in J$. The converse is also easy.

(ii) $\Rightarrow$ (i): (I) Assume $\mathcal{T} = (T^n)_{n=1}^{\infty}$ to be discrete. Let $0 \neq h \in L^1_+$ be a non-trivial lower-bound function for T. Denote, as usual,

$$L^1_0 := \{f \in L^1 : \|f_+\| = \|f_-\|\}.$$

Since h is a non-trivial lower-bound function, we obtain

$$\begin{aligned}\limsup_{n\to\infty} \|(\mathcal{A}^T_n f - h)_+\| &\leq 1 - \|h\| \\ &< 1 \qquad (\forall f \in \mathcal{D}),\end{aligned}$$

and henceforth, due to Theorem 3.1.13, T is mean ergodic. Then there exists a T-invariant density, say u. Since $L^1 = L^1_0 \oplus \mathbb{R}\cdot u$, it is enough to show that

$$\lim_{n\to\infty} \|T^n f\| = 0 \qquad (\forall f \in L^1_0). \tag{3.6}$$

Notice that $(\|T^n f\|)_{n=1}^{\infty}$ is a monotone sequence since T is a contraction. Hence

$$\begin{aligned}\|f\| &\geq \lim_{n\to\infty} \|T^n f\| \\ &= \inf_n \|T^n f\| \quad (\forall f \in L^1).\end{aligned}$$

Now suppose that there exists $f \in L^1_0$ with $2\alpha := \lim_{n\to\infty} \|T^n f\| > 0$. Then

$$\begin{aligned}2\alpha &= \lim_{n\to\infty} \|T^n f\| \\ &= \lim_{n\to\infty} \|T^n(f_+ - f_-)\| \\ &= \lim_{n\to\infty} \Big\|(T^n f_+ - \alpha h)_+ - (T^n f_- - \alpha h)_+\Big\| \\ &\leq \lim_{n\to\infty} \Big(\|(T^n f_+ - \alpha h)_+\| + \|(T^n f_- - \alpha h)_+\|\Big) \\ &= 2\alpha(1 - \|h\|),\end{aligned}$$

which is impossible. Notice that the inequality above is true, because h is a lower-bound function and $\|f_+\| = \|f_-\| \geq \alpha$ holds. Consequently, the condition (3.6) holds.

(II) Now assume that $\mathcal{T} = (T_t)_{t\geq 0}$ is a semigroup of Markov operators, which is not necessarily continuous. We shall prove the implication (ii) $\Rightarrow$ (i) in this case. In this way, we reproduce the elegant arguments from [71, Proof of Theorem 7.4.1]. Take any $t_0 > 0$ and define $T = T_{t_0}$. Then h is a non-trivial lower-bound function for $(T^n)_{n=1}^{\infty}$. The first part of the proof implies that there exists a unique T-invariant density u such that

$$\lim_{n\to\infty} T^n f = u \qquad (\forall f \in \mathcal{D}). \tag{3.7}$$

Having shown that $T_t u = u$ for $t \in \{kt_0\}_{k=1}^{\infty}$, we now demonstrate that $T_t u = u$ for all $t \in \mathbb{R}_+$. Pick $t' > 0$, set $f' = T_{t'} u$, and note that

$$u = T^n u = T_{nt_0} u\,.$$

Therefore,

$$\begin{aligned} \|T_{t'}u - u\| &= \lim_{n\to\infty} \|T_{t'}u - u\| \\ &= \lim_{n\to\infty} \|T_{t'}(T_{nt_0}u) - u\| \\ &= \lim_{n\to\infty} \|T_{nt_0}(T_{t'}u) - u\| \\ &= \lim_{n\to\infty} \|T^n(T_{t'}u) - u\| \\ &= \lim_{n\to\infty} \|T^n f' - u\| \\ &= 0 \end{aligned}$$

by (3.7). Since t' is arbitrary, we have that u is $\mathcal{T}$-invariant.

Finally, to show (3.5), pick a density f. Then

$$t \to \|T_t f - u\| = \|T_t f - T_t u\|$$

is a non-increasing function. Pick a subsequence $t_n := nt_0$. We know from before that $\lim_{n\to\infty} \|T_{t_n} f - u\| = 0$, hence $\lim_{t\to\infty} \|T_t f - u\| = 0$. □

For various applications of Theorem 3.2.1, we refer the reader to Lasota–Mackey's book [71].

3.2.2 We call $h \in L^1_+$ a *mean lower-bound function* for a Markov operator T if

$$\lim_{n\to\infty} \left\| \left(h - \mathcal{A}_n^T f \right)_+ \right\| = 0 \qquad (\forall f \in \mathcal{D}).$$

Obviously, any lower-bound function is a mean lower-bound function.

Theorem 3.2.2 (Emel'yanov–Wolff). *Let T be a Markov operator in L^1. Then the following assertions are equivalent*:

(i) *there exists a density u such that*

$$\lim_{n\to\infty} \|\mathcal{A}_n^T f - u\| = 0 \qquad (\forall f \in \mathcal{D});$$

(ii) *there exists a non-trivial mean lower-bound function for T.*

Proof. (i) ⇒ (ii): Let u satisfy $\lim_{n\to\infty} \|\mathcal{A}_n^T f - u\| = 0$ for all $f \in \mathcal{D}$, then u is a non-trivial mean lower-bound function for T.

(ii) ⇒ (i): Let $0 \neq h \in L^1_+$ be a non-trivial mean lower-bound function for T. Then

$$\limsup_{n\to\infty} \left\| (\mathcal{A}_n^T f - h)_+ \right\| \leq \eta \qquad (\forall f \in \mathcal{D}) \tag{3.8}$$

with $\eta := 1 - \|h\|$. By Theorem 3.1.13, T is mean ergodic, so we obtain a decomposition

$$L^1 = \mathrm{Fix}(T) \oplus \overline{(I - T)L^1}\,.$$

All that we need is that $\dim \mathrm{Fix}(T) = 1$, but this follows obviously from (3.8), since $\mathrm{Fix}(T)$ is a sublattice in L^1. □

A strongly continuous variant of Theorem 3.2.2 can be obtained similarly by using Theorem 3.1.13, or it can be derived from Theorem 3.2.2. It should be read as follows.

Theorem 3.2.3. *Let $\mathcal{T} = (T_t)_{t\geq 0}$ be a strongly continuous one-parameter Markov semigroup in L^1. Then the following assertions are equivalent*:

(i) *there exists a density u such that*

$$\lim_{t\to\infty} \left\| \frac{1}{t} \int_0^t T_\tau f d\tau - u \right\| = 0 \qquad (\forall f \in \mathcal{D});$$

(ii) *there exists $0 \neq h \in L^1_+$ satisfying*

$$\lim_{t\to\infty} \left\| \left(h - \frac{1}{t} \int_0^t T_\tau f d\tau \right)_+ \right\| = 0 \qquad (\forall f \in \mathcal{D}). \qquad \square$$

3.2.3 Now, we give a direct proof of the Komornik–Lasota theorem for a one-parameter semigroup of Markov operators. This theorem can be derived also from Theorem 2.2.5.

Theorem 3.2.4 (Komornik–Lasota). *Let $\mathcal{T}$ be a one-parameter Markov semigroup in an L^1-space. Then the following conditions are equivalent*:

(i) *the semigroup $\mathcal{T}$ is constrictive*;

(ii) *the semigroup $\mathcal{T}$ possesses a constrictor of the kind $[-y, y] + \eta B_{L^1}$ for some $\eta \in \mathbb{R}$, $0 \leq \eta < 1$;*

(iii) *there exist a positive $y \in L^1$ and $\eta \in \mathbb{R}$, $0 \leq \eta < 1$, such that*

$$\limsup_{t\to\infty} \|(T_t f - y)_+\| \leq \eta \qquad (\forall f \in \mathcal{D}).$$

Proof. Let us remark that it is enough to prove the theorem for a discrete semigroup only. So, let T be a Markov operator in an L^1-space X.

(i) ⇒ (ii): Let A be a compact constrictor for T. As any compact set, K can be included in the set $[-z, z] + 2^{-1} B_X$.

(ii) ⇔ (iii): It is trivial.

(iii) ⇒ (i): T is mean ergodic due to Theorem 3.1.13. Thus we may apply Theorem 2.1.8. This theorem allows us to assume

$$[-w, w] \in \mathrm{Constr}(T)$$

with some $w \in X_+$, $Tw = w$. Thus T is weakly almost periodic, since any order interval in an L^1-space is weakly compact. Applying Theorem 1.1.4 to T, we obtain the Markov projection P_r onto the space X_r of all reversible vectors of T. By the remark after Proposition 3.1.2, the space

$$Y := P_r(T)$$

is an L^1-space as a range of a Markov projection. Since the unit ball

$$B_Y = P_r(B_X) \subseteq [-w, w]$$

of the L^1-space Y is weakly compact, we obtain $\dim Y < \infty$, and hence the set $P_r(B_X)$ is compact.

We shall prove that $P_r(B_X) \in \mathrm{Constr}(T)$. It is enough to show that if 0 is in the weak closure of $\{T^n x\}_{n=0}^{\infty}$, then $\lim\limits_{t\to\infty} \|T_t x\| = 0$.

Now, we apply the ultra-filter technique from **1.3.16** and **1.3.17**. Let $\mathcal{U}$ be a free ultra-filter on $\mathbb{N}$ and let $X_{\mathcal{U}}$ be the ultra-power of X with respect to $\mathcal{U}$. Define

$$S : X \to X_{\mathcal{U}}, \;\; x \mapsto (T^n x)_{\mathcal{U}} =: S(x).$$

Since $[-w, w]$ is a constrictor of T, S maps B_X into $[-w, w]$ (where we have identified X with its canonical image in $X_{\mathcal{U}}$). Thus

$$S(X) \subseteq \mathcal{Y} := \bigcup_{n=1}^{\infty} \{x \in X_{\mathcal{U}} : |x| \leq n\, w\}.$$

We consider $\mathcal{Y}$ as a space equipped with the supremum-norm induced by $[-w, w]$. By the Krein–Kakutani representation theorem, $\mathcal{Y}$ is isomorphic to a $C(K)$-space and, in particular, $\mathcal{Y}$ possesses the Dunford–Pettis property. It is well known that $X_{\mathcal{U}}$ is an L^1-space as an ultra-power of the L^1-space X. Therefore, order intervals in $X_{\mathcal{U}}$ are weakly compact and the embedding $i : Y \to X_{\mathcal{U}}$ is weakly compact, and hence i maps weakly convergent sequences into norm convergent ones (cf. [110, II.9.7]).

Let $0 \in \text{w-cl}\{T^n x\}_{n=0}^{\infty}$, then, by the Eberlein–Smulian theorem (cf. [23, p.18]), there exists a subsequence $(T^{k_n} x)_{n=1}^{\infty}$ converging weakly to 0. Notice that $S = i \circ S_1$, where $S_1 : X \to Y$ is nothing else than S now viewed as a mapping to Y. S_1 is continuous and hence weakly continuous. So, we obtain

$$\begin{aligned} 0 = \lim_{n\to\infty} \|i(S_1(T^{k_n} x))\| &= \lim_{n\to\infty} \|S(T^{k_n} x)\| \\ &= \lim_{n\to\infty} \lim_{\mathcal{U}} \|T^m \circ T^{k_n}\, x\| \\ &= \lim_{n\to\infty} \|T^{m_n + k_n} x\| \end{aligned}$$

for an appropriate sequence $(m_n)_{n=1}^{\infty}$. Consequently, $\lim\limits_{n\to\infty} \|T^n x\| = 0$, since T is contractive. □

We leave the easy proof of the following corollary to the reader.

Corollary 3.2.5. *Any strongly continuous one-parameter Markov semigroup in an L^1-space possessing a constrictor $[-y, y] + \eta B_X$ for some $\eta \in \mathbb{R}$, $0 \le \eta < 1$, is strongly stable and has a finite rank limit-projection.* □

Related Results and Notes

3.2.4 Studying asymptotic behavior of Markov semigroups in this section, we have mostly avoided using integral type conditions like (11.5) or (11.10). Such type of conditions are visually more complicated and they cannot be used in Sections 15 and 16, where we shall investigate Markov semigroups in non-commutative L^1-spaces. However, conditions of integral type are more appropriate sometimes for concrete applications. Moreover, historically a lot of results on asymptotic behavior were obtained in integral form [62], [63], [64], [72], [121], etc. Here we briefly discuss integral conditions and refer for the more detailed explanation to [62] and [71].

The spectral decomposition theorem of Komornik and Lasota, that we have discussed above, was originally formulated in [64] in the integral form. It consists of two parts. The first part proves that the Markov semigroup, satisfying an integral condition which is equivalent to condition (iii) of Theorem 3.2.4, is constrictive and, henceforth, can be reduced asymptotically to the semigroup of permutation matrices. The second part is easy; it gives a concrete form of the semigroup of permutation matrices that was well known before [64] (see, for example, [110]). The non-trivial first part was proved above by use of a rather general and delicate Theorem 2.1.8. Here we shall do the rest of the work in proving Komornik–Lasota's theorem. We give the theorem for discrete Markov semigroups only. The strongly continuous case can be investigated in the same manner, and it is slightly easier due to Corollary 3.2.5.

Theorem 3.2.6 (Komornik–Lasota). *Let T be a Markov operator in $X = L^1(\Omega, \Sigma, \mu)$ satisfying the condition:*

> *There exist $C \in \Sigma$, $\mu(C) < \infty$, and constants $\kappa < 1$ and $\delta > 0$ with the property: for every $0 \le f \in L^1$, $\|f\| = 1$, there exists $n_0(f) \in \mathbb{N}$ such that*
>
> $$(\textstyle\int) \qquad \int_{(\Omega - C) \cup D} T^n f d\mu \le \kappa$$
>
> *for all $t \ge n_0(f)$, $\mu(D) \le \delta$.*

Then there are an integer r, two sequences of densities $g_i \in L^1$ and $k_i \in L^\infty$, $i = 1, \dots, r$, and operator $Q : L^1 \to L^1$ such that, for every $f \in L^1$, Tf may be written in the form

$$Tf(x) = \sum_{i=1}^{r} \lambda_i(f) g_i(x) + Qf(x),$$

where

$$\lambda_i(f) = \int_\Omega f(x)k_i(x)\mu(dx).$$

The functions g_i and operator Q have the following properties:

(i) $g_i(x)g_j(x) = 0$ *for all* $i \neq j$ *so that functions* g_i *have disjoint supports*;

(ii) *for each integer* i, *there exists the unique integer* $\sigma(i)$ *such that* $Tg_i = g_{\sigma(i)}$. *Further,* $\sigma(i) \neq \sigma(i)$ *for* $i \neq j$, *and thus operator* T *just serves to permute the functions* g_i;

(iii) $\lim_{n\to\infty} \|T^n \circ Q\, f\| = 0$ *for every* $f \in L^1$.

Proof. Applying Lemma 3.1.1 to the sequence $(T^n f)_{n=1}^\infty$ of densities, one may use Theorem 3.2.4, which implies that T is constrictive. So we have the decomposition $X = X_0(T) \oplus Y$ into two T-invariant subspaces such that

$$X_0(T) = \{x \in X : \lim_{n\to\infty} \|T^n x\| = 0\}$$

and Y is an L^1-space generated by the finite basis $\{e_k\}_{k=1}^{k=p}$ of pairwise disjoint densities. Operator T permutes $\{e_k\}_{k=1}^{k=p}$, while they are extreme points of the unit ball of Y, and the theorem follows. □

Let us remark that Theorem 3.2.6 easily follows from the more general result of Theorem 2.2.6.

3.2.5 Let us give an application of Theorem 3.2.2. As usual, let (Ω, Σ, μ) be a σ-finite measure space and let $K : \Omega \times \Omega \to \mathbb{R}_+$ be a Markov kernel. Define $T : L^1(\Omega, \Sigma, \mu) \to L^1(\Omega, \Sigma, \mu)$ by

$$Tf(x) = \int K(x,y)f(y)dy,$$

and by induction

$$K_1(x,y) = K(x,y)\,,$$
$$K_{n+1}(x,y) = \int K(x,z)K_n(z,y)dz\,.$$

Then $T^n f(x) = \int K_n(x,y)f(y)dy$ holds. Finally, we set

$$\overline{K}_n(x,y) = \frac{1}{n}\sum_{k=1}^{n} K_k(x,y).$$

Then

$$\begin{aligned}(T \circ \mathcal{A}_n^T)f(x) &= \frac{1}{n}\sum_{k=1}^{n} T^k f(x) \\ &= \int \overline{K}_n(x,y)f(y)dy\,.\end{aligned}$$

We obtain the following proposition.

Proposition 3.2.7. *Assume that*

$$\int \liminf_{n\to\infty} \left(\inf_y \overline{K}_n(x,y) \right) dx > 0.$$

Then T is mean ergodic and the space of its fixed vectors is one-dimensional.

Proof. Set

$$h(x) = \liminf_{n\to\infty} \left(\inf_y \overline{K}_n(x,y) \right),$$

then $h \neq 0$ by hypothesis. Moreover, an easy computation shows that h is a mean lower-bound function. Now it is enough to apply Theorem 3.2.2. □

Let (Ω, Σ, μ), $K(x,y)$, T, and $\overline{K}_n(x,y)$ be as before. Let $V : \Omega \to \mathbb{R}_+$ be an arbitrary measurable function which is not a null-function with respect to μ, and set

$$G_a := V^{-1}([0,a]) = \{x \in \Omega : V(x) \leq a\}.$$

Proposition 3.2.8. *Assume that there exist a constant $M > 0$ and a subset $\mathcal{D}_0$ of the set $\mathcal{D}$ of all densities which is dense in $\mathcal{D}$ such that*

$$\limsup_{n\to\infty} \int\int V(x)\overline{K}_n(x,y)f(y)dydx \leq M \quad \textit{for all}\, f \in \mathcal{D}_0.$$

Moreover, assume that for every $a > 0$ with $G_a \neq \emptyset$,

$$\int \inf_{y\in G_a} K(x,y)dx \ = \ \delta(a) > 0$$

holds. Then T is mean ergodic and the space of fixed vectors of T is one-dimensional.

Proof. Let $f \in \mathcal{D}_0$. Then there exists n_0 such that

$$\int\int V(x)\overline{K}_n(x,y)f(y)dydx \leq M + 1$$

for all $n \geq n_0$. Choose $a \geq 3M$ such that $G_a \neq \emptyset$. Then, by Chebyshev's inequality, we obtain for

$$C_n^T := \frac{1}{n}\sum_{k=1}^{n} T^k = T \circ \mathcal{A}_n^T$$

that

$$\begin{aligned} \int_{G_a} C_n^T f dx \ &\geq \ 1 - \frac{1}{a}\int V \circ C_n^T\, f\, dx \\ &> \ 1 - \frac{M+1}{3M} \\ &> \ 1/2 \end{aligned}$$

for $n \geq n_0$. Moreover,

$$C_{n+1}^T \geq \frac{n}{n+1} T \circ C_n^T$$

holds. Hence

$$\begin{aligned} C_{n+1}^T f(x) &\geq \frac{n}{n+1} \int K(x,y) C_n^T f(y) dy \\ &\geq \frac{n}{n+1} \int_{G_a} K(x,y) C_n^T f(y) dy \,. \end{aligned}$$

Therefore, we obtain for $h(x) = \inf_{y \in G_a} K(x,y)$

$$\begin{aligned} C_{n+1}^T f(x) &\geq \frac{n}{n+1} \inf_{y \in G_a} K(x,y) \int_{G_a} C_n^T f(y) dy \\ &\geq \frac{n}{n+1} \cdot \frac{1}{2} \cdot h(x) \\ &\geq \frac{h(x)}{4}. \end{aligned}$$

Thus

$$\lim_{n \to \infty} \|(C_n^T f - h/4)_-\| = 0$$

for all $f \in \mathcal{D}_0$. But since $\mathcal{D}_0$ is dense in $\mathcal{D}$, and $h \neq 0$ by hypothesis, $h/4$ is a mean lower-bound and the assertion follows from Theorem 3.2.2. □

Remark that if V is bounded, then the hypotheses on T implies that T dominates an operator $1_\Omega \otimes h$ from which it follows easily that h is a lower-bound function for T. Hence, by Theorem 3.2.1, T is not only mean ergodic but even asymptotically stable. So, Proposition 3.2.8 is only interesting in the case of an unbounded V (cf. also [71, Theorem 5.7.2]).

3.2.6 Let (X, Σ, μ) be a σ-finite measure space, $\phi : X \to X$ a non-singular transformation, and P the corresponding Frobenius–Perron operator. Operator P is said to *overlap supports of functions* if for every $f, g \in L^1(X, \Sigma, \mu)$, $f \geq 0$, $f \neq 0$, $g \geq 0$, $g \neq 0$, there exists a natural n (which depends on f and g) such that

$$P^n f \wedge P^n g \neq 0 \,.$$

Bartoszek and Brown [15] proved that if P overlaps supports of functions and if there exists an a.e.-positive P-invariant density, then P is asymptotically stable. Then Zaharopol [133] showed that it is enough to assume that there exists P-invariant density and that under this assumption P is asymptotically stable if and only if P overlaps supports of functions. It is interesting that if we drop the assumption that P is a Frobenius–Perron operator and consider the Markov operators, then this result is not true, as is easy to see from Example 3.1.28. For another proof of this theorem as well as for related comments, we refer to Lin [77].

3.3 Positive semigroups in non-commutative L^1-spaces

The main objects in this section are Markov operators in the predual of a von Neumann algebra. Let $\mathcal{M}$ be a *von Neumann algebra* with its unique *predual* $\mathcal{M}_*$. A *Markov operator* in $\mathcal{M}_*$ is a linear operator that maps the set $\mathcal{S}(\mathcal{M}) \subseteq \mathcal{M}_*$ of all *normal states* in $\mathcal{M}$ into itself. This notion correlates with the classical notion of a Markov operator in a von Neumann algebra $\mathcal{M}$, which is a linear normal positive unital preserving mapping on $\mathcal{M}$. Namely, a Markov operator in $\mathcal{M}_*$ is the predual operator for a Markov operator in the von Neumann algebra $\mathcal{M}$. We do not use this duality and deal only with Markov operators in the predual of a von Neumann algebra. For terminology and notation concerning operator algebras, as well as for the basic results about them, we refer the reader to [93], [106], [107], and [127].

During the last two decades, several important results about the asymptotic behavior of semigroups of Markov operators in $L^1(\Omega, \Sigma, \mu)$ with σ-finite measure μ were established (see, for instance, the book of Lasota and Mackey [71] and the paper of Komornik [63] for a survey and applications). Some of these results are presented in Sections 7 and 8. However, the technique which is used for obtaining these results does not work in a non-commutative setting. Our main goal in the present section is to investigate non-commutative variants of these results as well as satisfactory methods for proving them.

We prove that any predual of a von Neumann algebra has a strongly normal positive cone. This allows us to apply theorems from Sections 4 and 5 to Markov semigroups in $\mathcal{M}_*$. We establish Theorem 3.3.5, which says that under a certain condition a semigroup of Markov operators is mean ergodic. This theorem will be used to obtain the lower-bounds criterion of statistical stability and ergodicity of semigroups of Markov operators and it will be used to prove Theorem 3.3.10. Then we prove Theorem 3.3.10 on compression of constrictors and investigate inheritance of mean ergodicity of Markov operators under taking a power and under asymptotic domination.

Given a faithful semi-finite trace τ on a von Neumann algebra $\mathcal{M}$, one may be interested in the asymptotic behavior of Markov semigroups in the non-commutative L^1-space $L^1(\mathcal{M}, \tau)$. For any $f \in L^1(\mathcal{M}, \tau)$, there is the uniquely defined normal linear functional

$$\mathcal{M} \ni a \mapsto \varphi_f(a) := \tau(fa)$$

on $\mathcal{M}$, and the mapping $f \mapsto \varphi_f$ is a linear bi-positive isometric surjection onto the predual $\mathcal{M}_*$ of $\mathcal{M}$ (see [114, Thm.14]). Therefore, the results below can be easily applied to Markov semigroups in $L^1(\mathcal{M}, \tau)$.

To avoid a duplication, we formulate all these results which hold in both discrete and strongly continuous cases in one way. All results of this section are taken from papers [32], [36], [37], [38], and [108]. Let us point out that some results below hold also for semigroups of positive operators in rather general classes of ordered Banach spaces, including preduals of von Neumann algebras as well as Banach lattices with order continuous norm. We refer for such generalizations to Chapter 2.

3.3.1 We begin with a geometric property of the predual of a von Neumann algebra, namely, we shall show that the self-adjoint part of the predual of a von Neumann algebra is ordered by a strongly normal cone. We need the following technical lemma.

Lemma 3.3.1. *Let $\mathcal{H}$ be a Hilbert space and $\mathcal{M}$ be a von Neumann algebra in $\mathcal{L}(\mathcal{H})$. Let S, T, $U \in \mathcal{M}_+$ satisfy*

$$0 \leq S \leq T + U = I.$$

Then, for every $\eta \in \mathcal{H}$, the inequality

$$|\,(S\eta \mid \eta) - (T^{1/2} \circ S \circ T^{1/2}\eta \mid \eta)\,| \leq 2 \cdot \|\eta\| \cdot (U\eta \mid \eta)^{1/2}$$

holds.

Proof. Consider the equality

$$\begin{aligned} |\,(S\eta \mid \eta) - (T^{1/2} \circ S \circ T^{1/2}\eta \mid \eta)\,| &= |\,\|S^{1/2}\eta\|^2 - \|S^{1/2} \circ T^{1/2}\eta\|^2\,| \\ &= (\|S^{1/2}\eta\| + \|S^{1/2} \circ T^{1/2}\eta\|) \\ &\cdot \ |\,\|S^{1/2}\eta\| - \|S^{1/2} \circ T^{1/2}\eta\|\,| \end{aligned}$$

and the inequality

$$\begin{aligned} |\,\|S^{1/2}\eta\| - \|S^{1/2} \circ T^{1/2}\,\eta\|\,|^2 &\leq \|S^{1/2} \circ (I - T^{1/2})\,\eta\|^2 \\ &= ((I - T^{1/2}) \circ S \circ (I - T^{1/2})\,\eta \mid \eta) \\ &\leq ((I - T^{1/2})^2\eta \mid \eta)\,. \end{aligned}$$

Since $0 \leq T \leq I$ implies $0 \leq I - T^{1/2} \leq I + T^{1/2}$, it follows that

$$(I - T^{1/2})^2 \leq I - T = U\,.$$

So, we obtain

$$|\,(S\eta \mid \eta) - (T^{1/2} \circ S \circ T^{1/2}\eta \mid \eta)\,| \leq 2 \cdot \|\eta\| \cdot (U\eta \mid \eta)^{1/2}. \qquad \square$$

3.3.2 Now we are in a position to show that the self-adjoint part of the predual of a von Neumann algebra is ordered by a strongly normal cone.

Theorem 3.3.2 (Emel'yanov–Wolff). *Let $\mathcal{M}$ be a von Neumann algebra and let $\mathcal{M}_{*sa}$ be the self-adjoint part of the predual $\mathcal{M}_*$ of $\mathcal{M}$. Then the cone $\mathcal{M}_{*+}$ of all normal positive linear functionals in $\mathcal{M}$ is strongly normal in $\mathcal{M}_{*sa}$.*

Proof. Let $0 \le \mu, \nu \in \mathcal{M}_*$ be arbitrary, and set $\rho = |\mu - \nu|$. Let $0 \le \chi \le \mu$ be given. Then $0 \le \chi \le \nu + \rho$. We set

$$\gamma = \|\nu + \rho\|^{-1} \quad \text{and} \quad \chi_1 = \gamma\chi, \quad \nu_1 = \gamma\nu, \quad \rho_1 = \gamma\rho.$$

Then $\|\nu_1 + \rho_1\| = 1$. Apply the GNS-representation of $\mathcal{M}$

$$\pi_\lambda : \mathcal{M} \to \mathcal{B}(\mathcal{H})$$

induced by λ, and let $\xi \in \mathcal{B}(\mathcal{H})$ be a normalized cyclic vector for π_λ. Then there exist S, T, U in the positive cone $\pi_\lambda(\mathcal{M})'_+$ of the commutant $\pi_\lambda(\mathcal{M})'$ of $\pi_\lambda(\mathcal{M})$ satisfying

$$0 \le S \le T + U = I$$

as well as

$$\begin{aligned} \chi_1(a) &= (\xi \mid S \circ \pi_\lambda(a)\,\xi)\,,\ \nu_1(a) \\ &= (\xi \mid T \circ \pi_\lambda(a)\,\xi)\,,\ \rho_1(a) \\ &= (\xi \mid U \circ \pi_\lambda(a)\,\xi) \end{aligned}$$

for all $a \in \mathcal{M}$. We set

$$\psi(a) = (\xi \mid T^{1/2} \circ S \circ T^{1/2} \circ \pi_\lambda(a)\,\xi)\,.$$

Then $0 \le \psi \le \nu_1$. Denote

$$V = S - T^{1/2} \circ S \circ T^{1/2}.$$

By Lemma 3.3.1, we obtain for $a \in \mathcal{M}_+$:

$$\begin{aligned} |\psi(a) - \chi_1(a)| &= |(V \circ \pi_\lambda(a)\,\xi \mid \xi)| \\ &= |(V \circ \pi_\lambda(a)^{1/2}\,\xi \mid \pi_\lambda(a)^{1/2}\,\xi)| \\ &\le 2 \cdot \|\pi_\lambda(a)^{1/2}\xi\| \cdot (U \circ \pi_\lambda(a)^{1/2}\xi \mid \pi_\lambda(a)^{1/2}\xi)^{1/2} \\ &\le 2 \cdot \|\pi_\lambda(a)^{1/2}\xi\| \cdot \rho_1(a)^{1/2}. \end{aligned}$$

Since $\psi - \chi_1$ is self-adjoint, and since the norm is determined on the unit ball of the self-adjoint part $\mathcal{M}_{sa}$ of $\mathcal{M}$, we obtain

$$\begin{aligned} |\psi(a) - \chi_1(a)| &\le 2((\rho_1(a_+))^{1/2} + (\rho_1(a_-))^{1/2}) \\ &\le 4\|\rho_1\|^{1/2} \qquad (\forall a \in \mathcal{M}). \end{aligned}$$

Dividing this inequality by γ, we get

$$\begin{aligned} |\gamma^{-1}\psi(a) - \chi(a)| &\le \frac{4}{\sqrt{\gamma}} \cdot \|\rho\|^{1/2} \\ &\le 4\sqrt{2 \cdot \|\nu + \mu\| \cdot \|\rho\|} \qquad\qquad (\forall a \in \mathcal{M}). \end{aligned}$$

Now $0 \le \gamma^{-1}\psi \le \nu$ by construction. So we obtain

$$\operatorname{dist}(\chi, [0,\nu]) \le 4\sqrt{2 \cdot \|\nu + \mu\| \cdot \|\nu - \mu\|}$$

and, by straightforward calculation, it follows that

$$\operatorname{dist}([0,\mu], [0,\nu]) \le 4\sqrt{2 \cdot \|\nu + \mu\| \cdot \|\nu - \mu\|},$$

which yields the strong normality of $\mathcal{M}_{*+}$. □

3.3.3 We shall use often the following result of Akemann [3, Thm. II.2(2)] (see also [127, Thm. 5.4, p. 149]).

Theorem 3.3.3 (Akemann). *Any order interval in the predual of a Neumann algebra is weakly compact.* □

This result and Theorem 3.3.2 imply the following theorem.

Theorem 3.3.4 (Emel'yanov–Wolff). *Let $\mathcal{M}$ be a von Neumann algebra and let $\mathcal{M}_{*sa}$ be the self-adjoint part of the predual $\mathcal{M}_*$ of $\mathcal{M}$. Then $\mathcal{M}_{*sa}$ is an ideally ordered Banach space.* □

3.3.4 Let us fix some necessary notions, which are not fixed above. Let $\mathcal{M}$ be a von Neumann algebra with predual $\mathcal{M}_*$ and dual $\mathcal{M}^*$. For $x \le y$ in $\mathcal{M}_{*sa}$, we denote by $[x, y]$ the *order interval* $\{z \in \mathcal{M}_{*sa} : x \le z \le y\}$, and by $B_{\mathcal{M}_{*sa}} = \{z \in \mathcal{M}_{*sa} : \|z\| \le 1\}$ the closed unit ball of $\mathcal{M}_{*sa}$. The algebra $\mathcal{M}$ is called *atomic* if every nonzero projection dominates a nonzero minimal projection. For example, the algebra $\mathcal{M} = \mathcal{B}(\mathcal{H})$ of all linear bounded operators in a Hilbert space $\mathcal{H}$ is atomic. An operator T in $\mathcal{M}_*$ is called *completely positive* if its adjoint T^* is completely positive in $\mathcal{M}$ (cf. [127, p.200]). A positive operator T in $\mathcal{M}_*$ is called a *Markov operator* whenever the unit $\mathbb{1}$ of $\mathcal{M}$ is a fixed point of its adjoint T^*. Notice that, for a Markov operator T, the relations $\|T\| = 1$ and $\|Tf\| = \|f\|$ for $f \in (\mathcal{M}_*)_+$ hold.

Below let $\mathcal{T} = (T_t)_{t \in J}$ be a one-parameter discrete or strongly continuous semigroup of Markov operators in $\mathcal{M}_*$. The following theorem gives the conditions which ensure that a Markov semigroup in $\mathcal{M}_*$ is mean ergodic. For this result in the commutative case, we refer to Section 3.1.

Theorem 3.3.5 (Emel'yanov–Wolff). *Let $\mathcal{M}$ be a von Neumann algebra, let $\mathcal{T}$ be a one-parameter Markov semigroup in $\mathcal{M}_*$, $g \in \mathcal{M}_{*+}$, and let $\eta \in \mathbb{R}$, $0 \le \eta < 1$, such that*

$$\limsup_{t\to\infty} \operatorname{dist}(\mathcal{A}_t^{\mathcal{T}} f, [-g, g]) \le \eta$$

for any normal state $f \in \mathcal{M}_*$. Then $\mathcal{T}$ is mean ergodic.

If, moreover, $\mathcal{M}$ is atomic and $\mathcal{T}$ consists of completely positive operators, then the space $\operatorname{Fix}(\mathcal{T})$ *of all fixed vectors of $\mathcal{T}$ is finite-dimensional.*

Proof. First of all, we show that $\mathcal{T}$ is mean ergodic. By Theorem 1.1.11 and simple linearity arguments, it is enough to check that for every nonzero $\mathcal{T}^*$-fixed point $\psi \in \mathcal{M}_{sa}$ there exists a $\mathcal{T}$-fixed point $w \in \mathcal{M}_{*sa}$ satisfying $\langle \psi, w \rangle \neq 0$.

Let $\mathcal{M}_{sa} \ni \psi \neq 0$ be a fixed point of $\mathcal{T}^*$. We may assume that $\|\psi_+\| = \|\psi\| = 1$. Set $\varepsilon := (1-\eta)/3$ and take some $x \in \mathcal{M}_{*sa}$ which satisfies $\|x\| = 1$ and $\langle \psi_+, x \rangle \geq 1 - \varepsilon$. We have $\||x|\| = \|x\| = 1$ and

$$1 \geq \langle |\psi|, |x| \rangle \geq \langle \psi_+, |x| \rangle \geq \langle \psi_+, x \rangle \geq 1 - \varepsilon.$$

Thus

$$\begin{aligned} \langle \psi, |x| \rangle &= \langle 2\psi_+, |x| \rangle - \langle |\psi|, |x| \rangle \\ &\geq 2(1-\varepsilon) - 1 \\ &= 1 - 2\varepsilon\,. \end{aligned}$$

Let $x'' \in \mathcal{M}^*$ be a w^*-cluster point of $\{\mathcal{A}_t^{\mathcal{T}} |x|\}_{t \in J}$. Then $T_t^{**} x'' = x''$. Since

$$\limsup_{t \to \infty} \operatorname{dist}(\mathcal{A}_t^{\mathcal{T}} |x|, [-g, g]) \leq \eta$$

and $[-g, g]$ is weakly compact in $\mathcal{M}_*$ by Theorem 3.3.3, we obtain

$$x'' \in [-g, g] + \eta B_{\mathcal{M}^*} \subseteq \mathcal{M}_* + \eta B_{\mathcal{M}^*}.$$

Take the positive projection $R : \mathcal{M}^* \to \mathcal{M}_*$ according to [106, Prop.1.17.7]. Then

$$(Id_{\mathcal{M}^*} - R) x'' \in \eta B_{\mathcal{M}^*}\,,$$

and

$$\begin{aligned} \langle \psi, Rx'' \rangle &= \langle \psi_+, Rx'' \rangle - \langle \psi_-, Rx'' \rangle \\ &= \langle x'', \psi_+ \rangle - \langle (Id_{\mathcal{M}^*} - R) x'', \psi_+ \rangle - \langle \psi_-, Rx'' \rangle \\ &\geq \langle x'', \psi \rangle - \eta \\ &= \langle \psi, |x| \rangle - \eta \\ &\geq 1 - 2\varepsilon - \eta \\ &= \varepsilon \\ &> 0\,. \end{aligned}$$

Moreover,

$$\begin{aligned} T_s \circ R\,(x'') &= T_s \circ R \circ T_t^{**}\,(x'') \\ &\geq T_s \circ R \circ T_t^{**} \circ R\,(x'') \\ &= T_s \circ R \circ T_t \circ R\,(x'') \\ &= T_{s+t} \circ R\,(x'') \\ &\geq 0\,. \end{aligned}$$

Thus the net $(T_t \circ R\,x'')_{t\in J}$ is decreasing in $\mathcal{M}_{*+}$, and hence

$$w := \lim_{t\to\infty} T_t \circ R\,x''$$

exists. Clearly $T_t w = w$ for all $t \in J$, and

$$\langle \psi, w\rangle = \langle \psi, Rx''\rangle > 0\,.$$

Thus $\mathcal{T}$ is mean ergodic.

Now let $\mathcal{M}$ be atomic and let $\mathcal{T}$ be completely positive. The mean ergodic projection $P_{\mathcal{T}} = \lim_{t\to\infty} \mathcal{A}_t^{\mathcal{T}}$ is a completely positive Markov operator. By the theorem of Choi and Effros [20], the range of its adjoint $P'_{\mathcal{T}}$ is a von Neumann algebra, say $\mathcal{N}$. Hence the range

$$P_{\mathcal{T}}(\mathcal{M}_*) = \mathrm{Fix}(\mathcal{T})$$

is itself the predual of $\mathcal{N}$. Then the real part of the unit ball $B_{\mathcal{N}_{*sa}}$ of $\mathcal{N}_*$ satisfies

$$B_{\mathcal{N}_{*sa}} \subseteq [-g, g] + \eta B_{\mathcal{N}_{*sa}},$$

and hence

$$B_{\mathcal{N}_{*sa}} \subseteq \frac{1}{1-\eta}[-g, g].$$

Since order intervals of $\mathcal{M}_{*sa}$ are compact, the last inclusion shows that the unit ball of $\mathcal{N}_*$ is compact, and hence $\dim \mathrm{Fix}(\mathcal{T}) < \infty$. □

3.3.5 A Markov semigroup $\mathcal{T}$ in $\mathcal{M}_*$ is called *asymptotically stable* if there exists a normal state $u \in \mathcal{S}(\mathcal{M})$ such that

$$\lim_{t\to\infty} \|T_t f - u\| = 0 \qquad (\forall f \in \mathcal{S}(\mathcal{M})).$$

Such a normal state u is obviously unique and $\mathcal{T}$-invariant.

An element $h \in \mathcal{M}_{*+}$ is called a *lower-bound element* for $\mathcal{T}$ if

$$\lim_{t\to\infty} \|(h - T_t f)_+\| = 0 \qquad (\forall f \in \mathcal{S}(\mathcal{M})).$$

The following result is well known and due to Lasota (cf. Theorem 3.2.1) for Markov semigroups in a commutative L^1-space. Its generalization to the predual of a non-commutative von Neumann algebra is due to Sarymsakov and Grabarnik and it was announced without proof by Ayupov and Sarymsakov [14] and by Sarymsakov and Grabarnik in [108]. Indeed, it is a simple corollary of Theorem 3.3.5 above.

Theorem 3.3.6 (Ayupov–Sarymsakov–Grabarnik). *Let $\mathcal{M}$ be a von Neumann algebra. Then, for any one-parameter Markov semigroup $\mathcal{T}$ in $\mathcal{M}_*$, the following assertions are equivalent*:

(i) $\mathcal{T}$ *is asymptotically stable*;

(ii) *there exists a non-trivial lower-bound element for* $\mathcal{T}$.

Proof. (i) $\Rightarrow$ (ii): Let a normal state $u \in \mathcal{S}(\mathcal{M})$ satisfy

$$\lim_{t\to\infty} \|T_t f - u\| = 0 \qquad (\forall f \in \mathcal{S}(\mathcal{M})).$$

Then u is a non-trivial lower-bound element for $\mathcal{T}$.

(ii) $\Rightarrow$ (i): Let $0 \neq h \in \mathcal{M}_{*+}$ be a non-trivial lower-bound element for $\mathcal{T}$. Denote

$$\mathcal{M}_{*0} := \{f \in \mathcal{M}_{*sa} : \|f_+\| = \|f_-\|\}.$$

Since

$$\lim_{t\to\infty} \|(h - T_t f)_+\| = 0 \qquad (\forall f \in \mathcal{S}(\mathcal{M})),$$

then

$$\begin{aligned}\limsup_{t\to\infty} \|(\mathcal{A}_t^{\mathcal{T}} f - h)_+\| &\leq 1 - \|h\| \\ &< 1 \qquad (\forall f \in \mathcal{S}(\mathcal{M})).\end{aligned}$$

Theorem 3.3.5 applied to the interval

$$[-g, g] = [-h, h]$$

and

$$\eta = 1 - \|h\|$$

implies that $\mathcal{T}$ is mean ergodic, and hence there exists a $\mathcal{T}$-invariant normal state, say u. Obviously,

$$\mathcal{M}_{*sa} = \mathcal{M}_{*0} \oplus \mathbb{R} \cdot u$$

since $\mathcal{M}_{*0}$ has co-dimension 1 in $\mathcal{M}_{*sa}$.

Show the following:

$$\lim_{t\to\infty} \|T_t f - u\| = 0 \qquad (\forall f \in \mathcal{S}(\mathcal{M})).$$

It is enough to prove that

$$\lim_{t\to\infty} \|T_t f\| = 0 \qquad (\forall f \in \mathcal{M}_{*0}). \tag{3.9}$$

Note that $\|T_t f\| \geq \|T_{s+t} f\|$ since $\mathcal{T}$ is contractive. Hence

$$\|f\| \geq \lim_{t\to\infty} \|T_t f\| = \inf_t \|T_t f\|$$

holds for every f. If (3.9) fails, then there exists $f \in \mathcal{M}_{*0}$ with

$$2\alpha := \lim_{t\to\infty} \|T_t f\| > 0.$$

Then

$$\begin{aligned} 2\alpha &= \lim_{t\to\infty} \|T_t f\| \\ &= \lim_{t\to\infty} \|T_t(f_+ - f_-)\| \\ &= \lim_{t\to\infty} \|(T_t f_+ - \alpha h)_+ - (T_t f_- - \alpha h)_+\| \\ &\le \lim_{t\to\infty} (\|(T_t f_+ - \alpha h)_+\| + \|(T_t f_- - \alpha h)_+\|) \\ &= 2\alpha(1 - \|h\|), \end{aligned}$$

that is impossible. Here we use that

$$\lim_{t\to\infty} \|T_t f_+ - \alpha h - (T_t f_+ - \alpha h)_+\| = 0,$$

and

$$\lim_{t\to\infty} \|T_t f_- - \alpha h - (T_t f_- - \alpha h)_+\| = 0,$$

since h is a lower-bound element for $\mathcal{T}$. The contradiction shows that (3.9) holds. □

3.3.6 We call $h \in \mathcal{M}_{*+}$ a *mean lower-bound element* for a Markov semigroup $\mathcal{T}$ if

$$\lim_{t\to\infty} \|(h - \mathcal{A}_t^{\mathcal{T}} f)_+\| = 0 \qquad (\forall f \in \mathcal{S}(\mathcal{M})).$$

Obviously, any lower-bound element is a mean lower-bound element. Our next result is another application of Theorem 3.3.5, which is an analogue of Theorem 3.2.2.

Theorem 3.3.7 (Emel'yanov–Wolff). *Let $\mathcal{M}$ be a von Neumann algebra and let $\mathcal{T}$ be a one-parameter semigroup of completely positive Markov operators in $\mathcal{M}_*$. Then the following assertions are equivalent*:

(i) *there exists a $\mathcal{T}$-invariant normal state u such that*

$$\lim_{t\to\infty} \|\mathcal{A}_t^{\mathcal{T}} f - u\| = 0$$

for any normal state $f \in \mathcal{M}_*$;

(ii) *there exists a non-trivial mean lower-bound element for $\mathcal{T}$.*

Proof. (i) ⇒ (ii): Let $u \in \mathcal{M}_{*+}$ be such that $\lim_{t\to\infty} \|\mathcal{A}_t^{\mathcal{T}} f - u\| = 0$ for every normal state f, then u is a non-trivial mean lower-bound element for $\mathcal{T}$.

(ii) ⇒ (i): Let $0 \neq h \in \mathcal{M}_{*+}$ be a non-trivial mean lower-bound element for $\mathcal{T}$. Then

$$\limsup_{t\to\infty} \|(\mathcal{A}_t^{\mathcal{T}} f - h)_+\| \le \eta \qquad (\forall f \in \mathcal{S}(\mathcal{M}))$$

with

$$\eta := 1 - \|h\| .$$

By Theorem 3.3.5, $\mathcal{T}$ is mean ergodic. Thus, we only have to prove that the space $\mathrm{Fix}(\mathcal{T})$ is one-dimensional. Let P be the projection onto $\mathrm{Fix}(\mathcal{T})$ given by

$$Pf = \lim_{t\to\infty} \mathcal{A}_t(\mathcal{T})f .$$

Since h is a mean lower-bound element, we obtain

$$f = Pf \geq h$$

for all normal states $f \in \mathrm{Fix}(\mathcal{T}) \cap \mathcal{S}(\mathcal{M})$. But this implies $f \geq Ph =: h_0$ for all these normal states, and $h_0 \neq 0$ since

$$\begin{aligned} \|h_0\| &= \|Ph\| \\ &= \|\lim_{t\to\infty} \mathcal{A}_t(\mathcal{T})h\| \\ &= \|h\| \\ &> 0 . \end{aligned}$$

Now P is a completely positive Markov operator. Hence, by Choi–Effros's theorem used above in the proof of Theorem 3.3.5, $P(\mathcal{M}_*)$ is isometrically and order isomorphic to the predual $\mathcal{N}_*$ of a von Neumann algebra $\mathcal{N}$. Then every positive element $f \in \mathcal{N}_*$ of norm 1 dominates the nonzero element h_0, which obviously implies $\dim(\mathcal{N}_*) = 1$. □

3.3.7 Now we discuss the notion of constrictor for a one-parameter positive semigroup in the predual of a von Neumann algebra. Then we apply the results of Section 2.1 to positive semigroups in preduals of von Neumann algebras.

Let $\mathcal{M}$ be a von Neumann algebra with the predual $\mathcal{M}_*$, and let $\mathcal{T} = (T_t)_{t\in J}$ be a semigroup in $\mathcal{M}_*$. Given a non-empty subset $A \subseteq \mathcal{M}_{*sa}$ and a real $\alpha \geq 0$, we call A an *α-constrictor* for the semigroup $\mathcal{T}$ if

$$\limsup_{t\to\infty} \mathrm{dist}(T_t x, A) \leq \alpha \qquad (\forall x \in \mathcal{M}_{*sa}, \|x\| \leq 1).$$

This means exactly that the set $A + B_{\mathcal{M}_*}$ is a constrictor for $\mathcal{T}$ in the sense of Section 1.3. We denote the set of all α-constrictors for $\mathcal{T}$ by $\mathrm{Constr}_\alpha(\mathcal{T})$. We call $\mathcal{T}$ *constrictive*, whenever $\mathcal{T}$ possesses a compact 0-constrictor. Theorem 1.3.3 can be given in our setting in the following form:

Theorem 3.3.8. *An operator semigroup $\mathcal{T}$ in $\mathcal{M}_*$ is constrictive if and only if there exists a decomposition*

$$\mathcal{M}_* := \mathcal{M}_*^0 \oplus \mathcal{M}_*^r$$

into $\mathcal{T}$-invariant subspaces $\mathcal{M}_*^0$, $\mathcal{M}_*^r$ such that

$$\mathcal{M}_*^0 = \{x \in \mathcal{M}_* : \lim_{t\to\infty} \|T_t x\| = 0\} \qquad \text{and} \qquad \dim(\mathcal{M}_*^r) < \infty.$$

The problem arises to find other, weaker, conditions under which $\mathcal{T}$ is constrictive. In the commutative setting, this problem was investigated in Chapters 1 and 2.

3.3.8 We begin with the following lemma which is an easy consequence of Theorem 3.3.5.

Lemma 3.3.9. *Let $\mathcal{T}$ be a one-parameter positive semigroup in $\mathcal{M}_*$ which possesses an η-constrictor $[-g, g]$ for some real η, $0 \leq \eta < 1$. Then $\mathcal{T}$ is mean ergodic.*

Proof. It can be easily and directly checked that the semigroup $\mathcal{T}$ satisfies all conditions in Theorem 3.3.5 and, henceforth, is mean ergodic. □

Theorem 3.3.10 (Emel'yanov–Wolff). *Let $\mathcal{M}$ be a von Neumann algebra and $\mathcal{T} = (T_t)_{t\in J}$ be a one-parameter positive semigroup in $\mathcal{M}_*$. Assume that $\mathcal{T}$ possesses an η-constrictor $[-y, y]$ for some $0 \leq \eta < 1$ and some $y \in \mathcal{M}_+$. Then there exists a limit*

$$w := \lim_{\tau\to\infty} \mathcal{A}_\tau^{\mathcal{T}} y\,,$$

and the set $\frac{1}{1-\eta}[-w, w]$ is a 0-constrictor for $\mathcal{T}$. In particular, $\mathcal{T}$ is weakly almost periodic.

Proof. By Lemma 3.3.9, the semigroup $\mathcal{T}$ is mean ergodic. This fact and Theorem 3.3.4 allow us to apply Theorem 2.1.8, which says that $\frac{1}{1-\eta}[-w, w]$ is a 0-constrictor for $\mathcal{T}$. □

Theorem 3.3.11 (Emel'yanov–Wolff). *Let $\mathcal{M}$ be an atomic von Neumann algebra and let $\mathcal{T}$ be a one-parameter Markov semigroup in $\mathcal{M}_*$. Then $\mathcal{T}$ is constrictive if and only if $\mathcal{T}$ possess an η-constrictor $[-y, y]$ for some $0 \leq \eta < 1$ and some $y \geq \mathcal{M}_{*+}$.*

Proof. The sufficiency follows directly from Theorem 3.3.10, since every order interval in the predual of an atomic von Neumann algebra is compact [127, Cor.5.11, p.156]. The necessity holds obviously for Markov semigroups in the predual of any (not necessarily atomic) von Neumann algebra. □

3.3.9 Recall that a Markov operator T in $\mathcal{M}_*$ is called *irreducible* if its adjoint T' in $\mathcal{M}$ does not possess $\sigma(\mathcal{M}, \mathcal{M}_*)$-closed invariant hereditary sub-cones other than $\{0\}$ or $\mathcal{M}_+$ (see [53, p.388]).

Theorem 3.3.12. *Let T be a completely positive Markov operator in the predual $\mathcal{M}_*$ of an atomic von Neumann algebra $\mathcal{M}$. Assume that T is irreducible and there exists a positive $y \in \mathcal{M}_*$ such that*

$$[-y, y] \in \mathrm{Constr}_\eta(T)$$

for some real η, $0 \leq \eta < 1$.

Then there exists a Markov operator Q of finite rank such that $Q^{p+1} = Q$ for some $p \in \mathbb{N}$, and the sequence $(T^n - Q^n)_{n=0}^{\infty}$ converges to 0 *in the strong operator topology.*

Proof. Theorem 3.3.11 implies that $\mathcal{T}$ is constrictive, so $\dim(\mathcal{M}_*^r) < \infty$, where we use the notation of Theorem 3.3.8. By [52], the *peripheral point spectrum*

$$\sigma(T) \cap \{\lambda \in \mathbb{C} : |\lambda| = 1\}$$

of T is a finite group. Hence T is periodic in $\mathcal{M}_*^r$, and it is enough to set $Q := T \circ P_r$, where P_r is the Jacobs–Deleeuw–Glicksberg projection onto $\mathcal{M}_*^r$. ☐

3.3.10 Let T be a bounded linear operator in $\mathcal{M}_*$. If a power T^m of T is mean ergodic, then by Proposition 1.1.3, T itself is mean ergodic. The following result follows directly from Theorems 2.1.14 and 3.3.2.

Theorem 3.3.13 (Emel'yanov–Wolff). *Let $\mathcal{M}$ be a von Neumann algebra and let T be a mean ergodic Markov operator in $\mathcal{M}_*$. Then T^m is mean ergodic for any $m \in \mathbb{N}$.* ☐

Before we give the next result, let us recall a definition from Section 2.1. Let S, T be operators in $\mathcal{M}_*$. We say that S is *asymptotically dominated* by T if for any $f \in \mathcal{M}_{*+}$ there exists a sequence $(q_n^f)_{n=1}^{\infty} \subseteq \mathcal{M}_*$ such that

$$\lim_{n\to\infty} \|q_n^f\| = 0 \qquad \textit{and} \qquad T^n f + q_n^f \geq S^n f \quad (\forall n \in \mathbb{N}).$$

Theorem 3.3.14 (Emel'yanov–Wolff). *Let $\mathcal{M}$ be a von Neumann algebra and let S be a Markov operator in $\mathcal{M}_*$ which is asymptotically dominated by a* (*not necessarily linear*) *positive mean ergodic operator T. Then S is mean ergodic.*

Proof. By Theorem 3.3.2, $\mathcal{M}$ satisfies the conditions of Theorem 2.1.11. Applying of Theorem 2.1.11 finishes the proof. ☐

Every ordered Banach space E whose norm is additive on E_+ is uniformly order convex. In particular, this is the case for non-commutative L^1-spaces, and for the self-adjoint part of the dual of a C^*-algebra and self-adjoint part of the predual of a von Neumann algebra. For these spaces, we can apply directly the result of Theorem 2.1.17.

Related Results and Notes

3.3.11 In Theorems 3.3.11 and 3.3.12, we assumed $\mathcal{M}$ to be atomic. It is an open problem, whether these results hold for arbitrary von Neumann algebras. In this general setting, Theorem 3.3.10 shows immediately that $\mathcal{T}$ is weakly almost periodic. If, moreover, $\mathcal{T}$ consists of completely positive operators, then the space

$\mathcal{M}_*^r$ of reversible vectors is finite-dimensional, as follows easily from the Choi–Effros theorem mentioned above. The problem is whether the space $\mathcal{M}_*^f$ of all flight vectors coincides with

$$\mathcal{M}_*^0 = \{x : \lim_{t\to\infty} T_t x = 0\}\,.$$

Up to now, it is possible to show the equality of these two spaces only in the case above (whenever $\mathcal{M}$ is atomic) and in the case

$$\mathcal{M}_* = \mathcal{M}_{1*} \otimes \mathcal{M}_{2*}\,,$$

where $\mathcal{M}_1$ is commutative and $\mathcal{M}_2$ is finite-dimensional (see [37]).

3.3.12 We prove that the self-adjoint part of any C^*-algebra is also ordered by a strongly normal cone. Before we do this, we need the following lemma.

Lemma 3.3.15 (Emel'yanov–Wolff). *Let $\mathcal{A}$ be a C^*-subalgebra of $\mathcal{L}(\mathcal{H})$, where $\mathcal{H}$ is a Hilbert space. Let $0 \le S, T \in \mathcal{A}$ satisfy $S \le T + I$. Then there exists $V \in \mathcal{A}$ satisfying $0 \le V \le T$ and $\|S - V\| \le 2\sqrt{\|S\|}$. In particular, there holds*

$$\operatorname{dist}(S, [0,T] \cap \mathcal{A}) \le 2\sqrt{\|S\|}.$$

Proof. $0 \le S \le T + I$ implies

$$(T+I)^{-1/2} \circ S \circ (T+I)^{-1/2} \le I\,, \tag{3.10}$$

which in turn gives

$$T^{1/2} \circ (T+I)^{-1/2} \circ S \circ (T+I)^{-1/2} \circ T^{1/2} =: V \le T\,.$$

Obviously, $V \in \mathcal{A}$ whether or not I belongs to $\mathcal{A}$. Set

$$U = T^{1/2} \circ (T+I)^{-1/2}\,.$$

We show

$$\|S - V\| = \|S - U \circ S \circ U\| \le 2\sqrt{\|S\|}. \tag{3.11}$$

Since the operator

$$S - U \circ S \circ U$$

is self-adjoint, we obtain

$$\|S - U \circ S \circ U\| = \sup\{|(Sx|x) - (S \circ U\, x|U\, x)| : \|x\| = 1\}.$$

For $\|x\| = 1$, we have

$$\begin{aligned} |(Sx|x) - (S \circ U\, x|Ux)| &= |\,\|S^{1/2}x|^2 - \|S^{1/2} \circ U\, x\|^2\,| \\ &= (\|S^{1/2}\, x\| + \|S^{1/2} \circ U\, x\|) \\ &\quad \times |\,(\|S^{1/2}\, x\| - \|S^{1/2} \circ U\, x\|)\,|\,. \end{aligned}$$

Now $|\,\|S^{1/2}\,x\| - \|S^{1/2}\circ U\,x\|\,| \le \|S^{1/2}\circ (I-U)\,x\|$, and

$$\begin{aligned} \|S^{1/2}\circ (I-U)\,x\|^2 &= (S\circ (I-U)\,x|(I-U)\,x) \\ &= ((I-U)\circ S\circ (I-U)\,x|x). \end{aligned}$$

Finally,

$$\begin{aligned} &(I-U)\circ S\circ (I-U) \\ &= ((I+T)^{1/2}-T^{1/2})\circ (I+T)^{-1/2}\circ S\circ (I+T)^{-1/2}\circ ((I+T)^{1/2}-T^{1/2}) \end{aligned}$$

holds. Since the square root is monotone, we have $0\le (I+T)^{1/2}-T^{1/2}$. Using the inequality (3.10), we obtain

$$\begin{aligned} 0\le (I-U)\circ S\circ (I-U) &\le ((I+T)^{1/2}-T^{1/2})^2 \\ &\le I\,. \end{aligned}$$

Thus we have

$$|\,\|S^{1/2}x\| - \|S^{1/2}Ux\|\,| \le 1\,.$$

Using equality (3.11), we obtain

$$\begin{aligned} \|S-U\circ S\circ U\| &\le \|S^{1/2}\| + \|S^{1/2}\circ U\| \\ &\le \|S^{1/2}\|(1+\|U\|) \\ &\le 2\|S^{1/2}\| \\ &= 2\|S\|^{1/2}, \end{aligned}$$

and the lemma is proved. □

Theorem 3.3.16 (Emel'yanov–Wolff). *Let $\mathcal{A}$ be a C^*-algebra, and let $\mathcal{A}_{sa}$ be the self-adjoint part of $\mathcal{A}$. Then $\mathcal{A}_+$ is strongly normal in $\mathcal{A}_{sa}$.*

Proof. Without loss of generality one may assume that $\mathcal{A}$ is a C^*-subalgebra of the algebra $\mathcal{L}(\mathcal{H})$ of all bounded linear operators in the Hilbert space $\mathcal{H}$.

Let $0\le S,T\in\mathcal{A}$, and let $U=|S-T|$. We may assume that $U\ne 0$. Let $R\in\mathcal{A}$, $0\le R\le S$, be arbitrary. Then $0\le R\le T+U$, hence

$$0\le R\le T+\|U\|I\,.$$

Set $\gamma=\|U\|^{-1}$. Then Lemma 3.3.15 yields

$$\begin{aligned} \operatorname{dist}(R,[0,T]\cap\mathcal{A}) &= \gamma^{-1}\operatorname{dist}(\gamma R,[0,\gamma T]\cap\mathcal{A}) \\ &\le 2\gamma^{-1}\sqrt{\|\gamma R\|} \\ &\le 2\sqrt{\|U\|\,\|S\|}. \end{aligned}$$

This in turn implies

$$\operatorname{dist}([0,S]\cap\mathcal{A},[0,T]\cap\mathcal{A}) \le 2\sqrt{\|S\|+\|T\|}\sqrt{\|S-T\|},$$

which gives

$$\operatorname{dist}_H([0,S]\cap\mathcal{A},[0,T]\cap\mathcal{A})\le 2\sqrt{\|S\|+\|T\|}\sqrt{\|S-T\|},$$

and the assertion follows. □

Theorem 3.3.16 was used in 4.14 in the proof of Theorem 2.1.21.

3.3.13 There is another way to prove Theorem 3.3.6. We show how to do this now and give a little supplement to Theorem 3.3.6.

Theorem 3.3.17. *Let $\mathcal{T}=(T_t)_{t\in J}$ be a discrete or strongly one-parameter continuous semigroup of Markov operators in the predual $\mathcal{M}_*$ of the von Neumann algebra $\mathcal{M}$. Then the following assertions are equivalent*:

(i) *$\mathcal{T}$ is asymptotically stable*;

(ii) *there is $0\ne h\in(\mathcal{M}_*)_+$ such that for any $f\in(\mathcal{M}_*)_+$, $\|f\|=1$, and for any $t\in J$ there exists $f_t\in(\mathcal{M}_*)_+$ with $\lim_{t\to\infty}\|f_t\|=0$, and $T_tf+f_t\ge h$ for all $t\in J$*;

(iii) *$\mathcal{T}$ has a non-trivial lower-bound element.*

Proof. (i) ⇒ (iii): Let $0\ne u\in(\mathcal{M}_*)_+$ satisfy $\lim_{t\to\infty}T_tf=f(\mathbb{1})u$ for each $f\in\mathcal{M}_*$. Obviously, u is a non-trivial lower-bound element for $\mathcal{T}$.

(iii) ⇒ (ii): Let $0<h\in\mathcal{M}_*$ be a non-trivial lower-bound element for $\mathcal{T}$. Then, for any $f\in(\mathcal{M}_*)_+$, $\|f\|=1$, the condition (ii) is satisfied with

$$f_t:=(T_tf-h)_-$$

for all $t\in J$.

(ii) ⇒ (i): An easy computation shows that

$$\limsup_{t\to\infty}\operatorname{dist}(T_tf,[-h,h])\le 1-\|h\|$$

for all $f\in(\mathcal{M}_*)_{sa}$, $\|f\|\le 1$. Thus

$$[-h,h]+(1-\|h\|)B_{\mathcal{M}_*}\in\operatorname{Constr}_{1-\|h\|}(\mathcal{T})\,.$$

Applying Theorem 3.3.10 to $y=h$ and $\eta=1-\|h\|$, we obtain that

$$w=\lim_{\tau\to\infty}\mathcal{A}^{\mathcal{T}}_\tau h$$

exists and

$$\frac{1}{1-\eta}[-w,w]=\|h\|^{-1}[-w,w]$$

is a constrictor for $\mathcal{T}$. Moreover, $\|w\| = \|h\|$, since $\mathcal{T}$ consists of Markov operators. Put $u := \|h\|^{-1}w$, then $u \in (\mathcal{M}_*)_+$, $\|u\| = 1$ and $T_t u = u$ for all t. It follows directly from $[-u, u] \in \mathrm{Constr}_0(\mathcal{T})$ that

$$\lim_{t\to\infty} T_t f = u$$

for all $f \in (\mathcal{M}_*)_+$, $\|f\| = 1$ which is obviously equivalent to (i). □

3.3.14 The following result on operators in von Neumann algebras is a more or less direct consequence of results of Section 2.1 and Theorem 3.3.2.

Theorem 3.3.18. *Let $\mathcal{M}$ be a von Neumann algebra, $0 \leq S \leq T \in \mathcal{L}(\mathcal{M})$, and T a compact dual operator, then S is weakly compact. Moreover, if the algebra $\mathcal{M}$ is atomic, S is compact.*

Proof. Since $\mathcal{M}_{*sa}$ is an order ideal in $\mathcal{M}^*{}_{sa}$, the operator S itself is a dual operator, say $S = S_1{}'$. Denote also the predual for T by T_1.

It is enough to show that for every bounded sequence $(x_n)_{n=1}^\infty \subseteq \mathcal{M}^*{}_+$ there exists a subsequence $(x_{n_m})_{m=1}^\infty$ such that $(S_1 x_{n_m})_{m=1}^\infty$ is weakly convergent in $\mathcal{M}_*$.

Fix a sequence $(x_n)_{n=1}^\infty \subseteq \mathcal{M}^*{}_+$. Since T_1 is compact, there exists a convergent subsequence $(T_1 x_{n_k})_{k=1}^\infty$. Then

$$\{S_1 x_{n_k}\}_{k=1}^\infty \subseteq \bigcup_{k=1}^\infty [0, T_1 x_{n_k}],$$

and, henceforth, by Theorem 2.1.5, the set $\{S_1 x_{n_k}\}_{k=1}^\infty$ is conditionally weakly compact. Then there exists a weakly convergent subsequence $(S_1 x_{n_{k_i}})_{i=1}^\infty$, which implies that S_1 is weakly compact. It is known that, in this case, its dual S is weakly compact as well.

Whenever the von Neumann algebra $\mathcal{M}$ is atomic, each order interval in $\mathcal{M}_*$ is compact, and the same arguments with the use of Theorem 2.1.5 show that the sequence $(S_1 x_{n_m})_{m=1}^\infty$ has a norm convergent subsequence, and hence S_1 and S are both compact. □

Bibliography

[1] Yu.A. Abramovich, *Isometries of normed lattices,* Optimizatsiya **43(60)** (1988), 74–80.

[2] Yu.A. Abramovich; C.D. Aliprantis, *An invitation to operator theory,* Graduate Studies in Mathematics, **50** Amer. Math. Soc., Providence, RI, (2002).

[3] C.A. Akemann, *The dual space of an operator algebra,* Trans. Amer. Math. Soc. **126** (1967), 286–302.

[4] C.D. Aliprantis; O. Burkinshaw, *Positive Operators,* Academic Press, Orlando, London, (1985).

[5] C.D. Aliprantis; O. Burkinshaw, *Locally Solid Riesz Spaces with Applications to Economics,* Amer. Math. Soc. (2003).

[6] G.R. Allan; T.J. Ransford, *Power-dominated elements in a Banach algebra,* Studia Math. **94** (1989), no.1, 63–79.

[7] S. Alpay; A. Binhadjah; E.Yu. Emel'yanov, *A Positive Doubly Power Bounded Operator with a Nonpositive Inverse Exists on any Infinite-dimensional AL-Space,* Positivity **10** (2006), no.1, 105–110.

[8] S. Alpay; A. Binhadjah; E.Yu. Emel'yanov; Z. Ercan, *Mean Ergodicity of Positive Operators in KB-spaces,* J. Math. Anal. Appl. **323** (2006), 371–378.

[9] F. Andreu; J.M. Mazon, *On the boundary spectrum of dominated C_0-semigroups,* Semigroup Forum **38** (1989), 129–139.

[10] S.I. Ansari; P.S. Bourdon, *Some properties of cyclic operators,* Acta Sci. Math. (Szeged) **63** (1997), no.1-2, 195–207.

[11] W. Arendt; C.J.K. Batty, *Tauberian theorems for one-parameter semigroups,* Trans. Amer. Math. Soc. **306** (1988), 837–852.

[12] W. Arendt; C.J.K. Batty, *Domination and ergodicity for positive semigroups,* Proc. Amer. Math. Soc. **114** (1992) no. 3, 743–747.

[13] W. Arendt; C.J.K. Batty; M. Hieber; F. Neubrander, *Vector-valued Laplace transforms and Cauchy problems,* Monographs in Mathematics **96**. Basel: Birkhäuser (2001).

[14] Sh.A. Ayupov; T.A. Sarymsakov, *Markov operators on quantum probability spaces,* Probability theory and applications, Proc. World Congr. Bernoulli Soc., Tashkent/USSR 1986, Vol.1 (1987), 445–454.

[15] W. Bartoszek; T. Brown, *On Frobenius – Perron operators which overlap supports,* Bull. Polish Acad. Sci. Math. **45** (1997), 17–24.

[16] C.J.K. Batty, *On some ergodic properties for continuous and affine functions,* Ann. Inst. Fourier (Grenoble) **28** (1978), 209–215.

[17] C.J.K. Batty; D.W. Robinson, *Positive one-parameter semigroups on ordered Banach spaces,* Acta Applicandae **2** (1984), 221–296.

[18] A. Calderon, *Sur les measures invariantes,* C.R. Acad. Sci. Paris **240** (1955), 1960–1962.

[19] V. Caselles, *On the peripheral spectrum of positive operators,* Israel J. Math. **58** (1987), 144–160.

[20] M.D. Choi; E.G. Effros, *Injectivity and operator spaces,* J. Funct. Anal. **24** (1977), 156–209.

[21] M.M. Day, *On the basis problem in normed spaces,* Proc. Amer. Math. Soc. **13** (1962), 655–658.

[22] R. Datko, *Uniform asymptotic stability of evolutionary processes in a Banach space,* SIAM J. Math. Anal. **3** (1972), 428–445.

[23] J. Diestel, *Sequences and series in Banach spaces,* Graduate Texts in Mathematics, **92**. Springer-Verlag, New York - Berlin (1984).

[24] Y. Derriennic; U. Krengel, *Subadditive mean ergodic theorems,* Ergodic Theory Dynamical Systems **1** (1981), no.1, 33–48.

[25] Y.N. Dowker, *On measurable transformations in finite measure spaces,* Ann. of Math. (2) **62** (1955), 504–516.

[26] R. Drnovšek, *On positive unipotent operators on Banach lattices,* (preprint) (2006).

[27] E.Yu. Emel'yanov, *Banach lattices on which every power-bounded operator is mean ergodic,* Positivity **1** (1997), no.4, 291-296.

[28] E.Yu. Emel'yanov, *Invariant densities and mean ergodicity of Markov operators,* Israel J. Math. **136** (2003), 373–379.

[29] E.Yu. Emel'yanov, *A remark to a theorem of Yu. A. Abramovich,* Proc. Amer. Math. Soc. **132** (2004), no.3, 781–782.

[30] E.Yu. Emel'yanov, *Conditions for the regularity of Markov semigroups on abstract L^1-spaces,* Mat. Tr. **7** (2004), no.1, 50–82.

[31] E.Yu. Emel'yanov, *Conditions for a C_0-semigroup to be asymptotically finite-dimensional,* Siberian Math. J. **44** (2003), no.5, 793–796.

[32] E.Yu. Emel'yanov; U. Kohler; F. Räbiger; M.P.H. Wolff, *Stability and almost periodicity of asymptotically dominated semigroups of positive operators,* Proc. Amer. Math. Soc. **129** (2001), 2633–2642.

[33] E.Yu. Emel'yanov; F. Räbiger; M.P.H. Wolff, *Asymptotic behavior of positive operators on Banach lattices,* Positivity **4** (2000), no.3, 245–251.

[34] E.Yu. Emel'yanov; M.P.H. Wolff, *Mean ergodicity on Banach lattices and Banach spaces,* Arch. Math. (Basel) **72** (1999), no.3, 214–218.

[35] E.Yu. Emel'yanov; M.P.H. Wolff, *Quasi constricted linear operators on Banach spaces,* Studia Math. **144** (2001), no.2, 169–179.

[36] E.Yu. Emel'yanov; M.P.H. Wolff, *Asymptotic behaviour of Markov semigroup on noncommutative L_1-spaces,* Proceedings of the Conference "Quantum Probability and Infinite Dimensional Analysis" March 15–20, 2001 - Burg/Spreewald, (2001), Proceedings of the Conference "QP-PQ: Quantum Probability and White Noise Analysis", World Scientific, vol. XVI (2002).

[37] E.Yu. Emel'yanov; M.P.H. Wolff, *Positive operators on Banach spaces ordered by strongly normal cone,* Positivity **7** (2003), no.1–2, 3–22.

[38] E.Yu. Emel'yanov; M.P.H. Wolff, *Quasi constricted linear representations of abelian semigroups on Banach spaces,* Math. Nachr. **233**–**234** (2002), 103–110.

[39] E.Yu. Emel'yanov; M.P.H. Wolff, *Mean lower bounds for Markov operators,* Ann. Pol. Math. **83** (2004), no.1, 11–19.

[40] E.Yu. Emel'yanov; M.P.H. Wolff, *Asymptotic Behavior of Markov Semigroups on Preduals of von Neumann Algebras,* J. Math. Anal. Appl. **314** (2006), 749–763.

[41] K.-J. Engel; R. Nagel *One-parameter semigroups for linear evolution equations. With contributions by S. Brendle, M. Campiti, T. Hahn, G. Metafune, G. Nickel, D. Pallara, C. Perazzoli, A. Rhandi, S. Romanelli and R. Schnaubelt,* Graduate Texts in Mathematics, **194** Springer-Verlag, New York, (2000).

[42] J. Esterle, *Quasimultipliers, representations of H^∞, and the closed ideal problem for commutative Banach algebras,* (Radical Banach algebras and automatic continuity) (Long Beach, Calif., 1981), 66–162, Lecture Notes in Math., **975** Springer – Berlin (1983).

[43] S.R. Foguel, *The Ergodic Theory of Markov Processes,* Van Nostrand-Reinhold, New York (1969).

[44] S.R. Foguel; B. Weiss, *On convex power series of a conservative Markov operator,* Proc. Amer. Math. Soc. **38** (1973), 325–330.

[45] V.P. Fonf; M. Lin; P. Wojtaszczyk, *Ergodic Characterization of Reflexivity of Banach spaces,* J. Funct. Anal. **187** (2001), 146–162.

[46] I. Gelfand, *Zur Theorie der Charaktere der Abelschen topologischen Gruppen,* Rec. Math. [Mat. Sbornik] N.S. **9(51)**, (1941), 49–50.

[47] R. Godement, *Théorémes taubériens et théorie spectrale,* Ann. Sci. École Norm. Sup. (3) **64** (1947), 119–138.

[48] J.A. Goldstein, *Semigroups of linear operators and applications,* Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, (1985), x+245 pp.

[49] J.A. Goldstein, *Semigroups of linear and nonlinear operations and applications,* Kluwer Academic Publishers Group, Dordrecht, (1993), 9–57.

[50] S.G. Gorokhova; E.Yu. Emel'yanov, *A sufficient condition for the order boundedness of the attractor of a positive mean ergodic operator acting on a Banach lattice,* Siberian Adv. Math. **9** (1999), no.3, 78–85.

[51] A. Goullet de Rugy, *La structure idèale des M-spaces,* J. Math. Pures Appl. IX., Sè. **51** (1972), 331–373.

[52] U. Groh, *The peripheral point spectrum of Schwarz operators on C^*–algebras,* Math. Z. **176** (1981), 311–318.

[53] U. Groh, *Spectrum and asymptotic behaviour of completely positive maps on $\mathcal{B}(H)$,* Math. Japon. **29** (1984), no.3, 395–402.

[54] U. Groh; F. Neubrander, *Stabilität starkstetiger, positiver Operatorhalbgruppen auf C^*-Algebren,* Math. Ann. **256** no.4 (1981), 509–516.

[55] H. Helmberg, *On the converse of the Hopf ergodic theorem,* Zeitschr. Warscheinlichkeitsth. verw Gebiete **21** (1972), 165–179.

[56] E. Hille, *Functional Analysis and Semi-Groups,* Amer. Math. Soc. Colloq. Publ., vol.**31** Amer. Math. Soc., New York, (1948).

[57] E. Hille; R.S. Phillips, *Functional analysis and semi-groups. rev. ed.,* Amer. Math. Soc. Colloq. Publ., vol.**31** Amer. Math. Soc., New York, (1957).

[58] B. Jamison, *Irreducible Markov operators on $C(S)$,* Proc. Amer. Math. Soc. **24** (1970) 366–370.

[59] S. Karlin, *Positive operators,* J. Math. Mech. **8** (1959) 907–937.

[60] T. Kato, *Perturbation theory for linear operators,* Die Grundlehren der mathematischen Wissenschaften, Band **132** Springer-Verlag New York, Inc., New York (1966).

[61] Y. Katznelson; L. Tzafriri, *On power bounded operators,* J. Funct. Anal. **68** (1986), 313–328.

[62] J. Komornik, *Asymptotic periodicity of the iterates of weakly constrictive Markov operators,* Tohoku Math. J. (2) **38** no.1 (1986), 15–27.

[63] J. Komornik, *Asymptotic periodicity of Markov and related operators,* Dynamics reported, Dynam. Report. Expositions Dynam. Systems (N.S.) **2** (1993) 31–68, Springer, Berlin.

[64] J. Komornik; A. Lasota, *Asymptotic decomposition of Markov operators,* Bull. Polish Acad. Sci. Math. **35** (1987), no.5–6, 321–327.

[65] I. Kornfeld; M. Lin, *Weak almost periodicity of L_1-contractions and coboundaries of non-singular transformations,* Studia Math. **138** (2000), no.3, 225–240.

[66] M.G. Krein; M.A. Krasnoselskii; D.P. Milman, *On the defect numbers of linear operators in a Banach space and on some geometrical questions,* Sb. Tr. Inst. Mat. Akad. Nauk Ukr. SSR **11** (1948), 97–112.

[67] U. Krengel, *Ergodic Theorems,* De Gruyter, Berlin – New York (1985).

[68] J. Lamperti, *On the isometries of certain function-spaces,* Pacific J. Math. **8** (1958), 459–466.

[69] A. Lasota, *Statistical stability of deterministic systems,* Equadiff 82 (Wurzburg, 1982), 386–419, Lecture Notes in Math., **1017** Springer – Berlin (1983).

[70] A. Lasota; T.Y. Li; J.A. Yorke, *Asymptotic periodicity of the iterates of Markov operators,* Trans. Amer. Math. Soc. **286** (1984), 751–764.

[71] A. Lasota; M.C. Mackey, *Chaos, fractals, and noise. Stochastic aspects of dynamics,* Second edition. Applied Mathematical Sciences, **97**. Springer-Verlag, New York, (1994).

[72] A. Lasota; J. Socala, *Asymptotic properties of constrictive Markov operators,* Bull. Polish Acad. Sci. Math. **35** (1987), no.1–2, 71–76.

[73] R. deLaubenfels; Vu Quoc Phòng, *The discrete Hille-Yosida space and the asymptotic behaviour of individual orbits of linear operators,* J. Funct. Anal. **142** (1996), no. 2, 539–548.

[74] P.D. Lax, *Functional analysis,* Pure and Applied Mathematics (New York). Wiley-Interscience, New York, (2002).

[75] K. deLeeuw; I. Glicksberg, *The decomposition of certain group representations,* J. Analyse Math. **15** (1965), 135–192.

[76] M. Lin, *On the uniform ergodic theorem,* Proc. Amer. Math. Soc. **43** (1974), 337–340.

[77] M. Lin, *Support overlapping L^1 contractions and exact non-singular transformations,* Colloquium Math. **84/85** (2000), part.2, 515–520.

[78] J. Lindenstrauss; L. Tzafriri, *Classical Banach spaces,* Vol. I, Ergeb. Math. Grenzgeb., Vol.92, Springer, Berlin 1977.

[79] J. Lindenstrauss; L. Tzafriri, *Classical Banach spaces,* Vol. II, Ergeb. Math. Grenzgeb., Springer, Berlin 1979.

[80] Yu.I. Lyubich, *Introduction to the Theory of Banach Representations of Groups,* Birkhäuser: Basel, Boston, Berlin (1988).

[81] M.Yu. Lyubich; Yu.I. Lyubich, *The spectral theory of almost periodic representations of semigroups,* Ukrain. Mat. Zh. **36** (1984), no.5, 632–636.

[82] Yu.I. Lyubich; Vu Quoc Phòng, *Asymptotic stability of linear differential equations in Banach spaces,* Studia Math. **88** (1988), no.1, 37–42.

[83] Yu.I. Lyubich; Vu Quoc Phòng, *A spectral criterion for asymptotic almost periodicity for uniformly continuous representations of abelian semigroups,* J. Soviet Math. **49** (1990), no.6, 1263–1266

[84] J. Martinez; J.M. Mazon, *Quasi-compactness of dominated positive operators and C_0-semigroups,* Math. Z. **207** (1991), no.1, 109–120.

[85] P. Meyer-Nieberg, *Banach Lattices,* Universitext. Springer-Verlag, Berlin, (1991).

[86] V. Müller, *Local spectral radius formula for operators in Banach spaces,* Czechoslovak Math. J. **38(113)** (1988), no.4, 726–729.

[87] R. Nagel (ed.) *One-parameter semigroups of positive operators,* Lecture Notes in Mathematics **1184** Springer – Berlin (1986).

[88] J.M.A.M. van Neerven, *On the orbits of an operator with spectral radius one,* Czechoslovak Math. J. **45(120)** (1995), no.3, 495–502.

[89] J.M.A.M. van Neerven, *The asymptotic behaviour of semigroups of linear operators,* Operator Theory: Advances and Applications **88**. Birkhäuser, Basel, (1996).

[90] J.M.A.M. van Neerven, *Lower semicontinuity and the theorem of Datko and Pazy,* Integral Equations Operator Theory **42** (2002), no.4, 482–492.

[91] F. Neubrander, *Laplace transform and asymptotic behavior of strongly continuous semigroups,* Houston J. Math. **12** (1986), no.4, 549–561.

[92] B. de Pagter; A. Schep, *Measures of non-compactness of operators in Banach lattices,* J. Funct. Anal. **78** (1988), 31–55.

[93] G.K. Pedersen, *C^*-algebras and their automorphism groups,* Academic Press London New York San Francisco (1979).

[94] Vu Quoc Phòng, *Asymptotic almost periodicity and compactifying representations of semigroups,* Ukrain. Mat. Zh. **38** (1986), 688–692.

[95] Vu Quoc Phòng, *A short proof of the Y. Katznelson's and L. Tzafriri's theorem,* Proc. Amer. Math. Soc. **115** (1992), no.4, 1023–1024.

[96] Vu Quoc Phòng, *Stability of semigroups commuting with a compact operator,* Proc. Amer. Math. Soc. **124** (1996), no.10, 3207–3209.

[97] Vu Quoc Phòng, *Almost periodic and strongly stable semigroups of operators,* Banach Center Publ., Polish Acad. Sci., Warsaw, **38** (1997), 401–426.

[98] F. Räbiger, *Ergodic Banach lattices,* Indag. Math. (N.S.) **1** (1990), no.4, 483–488.

[99] F. Räbiger, *Stability and ergodicity of dominated semigroups: I. The uniform case,* Math. Z. **214** (1993), 43–54.

[100] F. Räbiger, *Stability and ergodicity of dominated semigroups: II. The strong case,* Math. Ann. **297** (1993), 103–116.

[101] F. Räbiger, *Attractors and asymptotic periodicity of positive operators on Banach lattices,* Forum Math. **7** (1995), 665–683.

[102] F. Räbiger; M.P.H. Wolff, *Spectral and asymptotic properties of dominated operators,* J. Austral. Math. Soc. (Series A) **63** (1997), 16–31.

[103] F. Räbiger; M.P.H. Wolff, *On the approximation of positive operators and the behaviour of the spectra of the approximants,* Integral Equations Operator Theory **28** (1997), no.1, 72–86.

[104] D.W. Robinson, *Basic theory of one-parameter semigroups,* Proc. Centre Math. Anal., Australian National University, Canberra, (1982).

[105] W. Rudin, *Functional analysis. Second edition,* International Series in Pure and Applied Mathematics. McGraw-Hill, Inc., New York, (1991).

[106] S. Sakai, *C^*-algebras and W^*-algebras,* Ergebnisse der Mathematik und ihrer Grenzgebiete, Band **60**. Springer-Verlag, New York – Heidelberg (1971).

[107] T.A. Sarymsakov, *Introduction to quantum probability theory,* Fan, Tashkent, (1985).

[108] T.A. Sarymsakov; T.Ya. Grabarnik, *The regularity of monotone continuous compresions on von Neumann algebras,* Doklady AN Uz.SSR **6** (1987), 9–11.

[109] S. Saxon; M. Levin, *Mark Every countable-codimensional subspace of a barrelled space is barrelled,* Proc. Amer. Math. Soc. **29** (1971), 91–96.

[110] H.H. Schaefer, *Banach Lattices and Positive Operators,* Die Grundlehren der mathematischen Wissenschaften, Band **215**. Springer-Verlag, New York – Heidelberg, (1974).

[111] H.H. Schaefer, *The zero-two law for positive contractions is valid in all Banach lattices,* Israel J. Math. **59** (1987), no.2, 241–244.

[112] H.H. Schaefer (with M.P.H. Wolff), *Topological Vector Spaces,* 2nd ed. Graduate Texts in Mathematics, **3**. Springer-Verlag, New York, (1999).

[113] H.H. Schaefer; M.P.H. Wolff; W. Arendt, *On lattice isomorphisms with positive real spectrum and groups of positive operators,* Math. Z. **164** (1978), no.2, 115–123.

[114] I.E. Segal *A non-commutative extension of abstract integration,* Annals Math. **57** (1953), 401–457.

[115] S.Y. Shaw, *Uniform ergodic theorems for locally integrable semigroups and pseudoresolvents,* Proc. Amer. Math. Soc. **98** (1986), no.1, 61–67.

[116] S.Y. Shaw, *Abstract ergodic theorems,* Proceedings of the Third International Conference on Functional Analysis and Approximation Theory, Vol. I (Acquafredda di Maratea, 1996). Rend. Circ. Mat. Palermo (2) Suppl. No.52, Vol.I (1998), 141–155.

[117] R. Sine, *A mean ergodic theorem,* Proc. Amer. Math. Soc. **24** (1970), no.2, 438–439.

[118] R. Sine, *A note on the ergodic properties of homeomorphisms,* Proc. Amer. Math. Soc. **57** (1976), no.1, 169–172.

[119] R. Sine, *Weakly constricted operators and Jamison's convergence theorem,* Proc. Amer. Math. Soc. **106** (1989), no.3, 751–755.

[120] R. Sine, *Constricted systems,* Rocky Mountain J. Math. **21** (1991), 1373–1383.

[121] J. Socala, *On the existence of invariant densities for Markov operators,* Ann. Polon. Math. **48** (1988), no.1, 51–56.

[122] K.V. Storozhuk, *Stabilizability in asymptotically finite-dimensional semigroups,* Siberian Math. J. **44** (2003), no.6, 1365–1376.

[123] K.V. Storozhuk, *On the Rolewicz theorem for evolution operators,* Proc. Amer. Math. Soc. (to appear) (2006).

[124] K.V. Storozhuk, *An extension of the Vu – Sine theorem and compact supercyclicity,* (preprint) (2007).

[125] E. Straube, *On the existence of invariant, absolutely continuous measures,* Comm. Math. Phys. **81** (1981), no.1, 27–30.

[126] L. Sucheston, *Problems, Probability in Banach Spaces,* Oberwolfach (1975), Lecture Notes in Math. **526** 285–289 Springer – Berlin – Heidelberg – New York (1976).

[127] M. Takesaki, *Theory of Operator Algebras I,* Springer-Verlag, New York – Heidelberg, (1979).

[128] A.I. Veksler; V.A. Geĭler, *Order and disjoint completeness of linear partially ordered spaces,* Siberian. Math. J. **13** (1972), no.1, 30–35.

[129] Yau-Chuen Wong; Kung-Fu Ng, *Partially Ordered Toplogical Vector Spaces,* Clarendon Press: Oxford (1973).

[130] K. Yosida, *Functional analysis,* Second edition. Classics in Mathematics. Springer-Verlag, Berlin (1968).

[131] A.C. Zaanen, *Riesz Spaces II,* North-Holland: Amsterdam (1983).

[132] R. Zaharopol, *Mean Ergodicity of Power-Bounded Operators in Countably Order Complete Banach Lattices,* Math. Z. **192** (1986), no.1, 81–88.

[133] R. Zaharopol, *Strongly asymptotically stable Frobenius-Perron operators,* Proc. Amer. Math. Soc. **128** (2000) no.12, 3547–3552.

[134] Xiao-Dong Zhang, *On spectral properties of positive operators,* Indag. Math. (N.S.) **4** (1993), no.1, 111–127.

[135] J. Zemanek, *On the Gelfand – Hille theorems,* Functional analysis and operator theory (Warsaw, 1992), 369–385, Banach Center Publ., **30** Polish Acad. Sci. – Warsaw (1994).

Index

List of special symbols